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TOWARD A GENERAL THEORY OF ELASTIC EQUILIBRIUM EQUATIONS FOR AN ISOTROPIC BODY

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TOWARD A GENERAL THEORY OF ELASTIC EQUILIBRIUM EQUATIONS FOR AN ISOTROPIC BODY

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ABSTRACT. The elastic equilibrium problem for bodies limited by planes and spheres is treated using the theory of Green's functions and harmonic functions for the case of given stresses and displacements on the boundary, as well as for the mixed problem.

REPORT II /13*

(Bodies Limited by Two Parallel Planes or Two Concentric Spheres)

I. Problems in Which the Elastic Body is Limited by Two Parallel Planes**

1. *Green Functions and Harmonic Functions.* Let us assume that the two planes limiting the elastic body are the two planes z = 0 and z = h, which we will indicate by σ_1 and σ_2, respectively, while we will continue to denote the combination of σ_1 and σ_2 by σ. The portion of space limited by them also retains its notation, S.

If $A \equiv (x, y, z)$ is a point in S -- i.e., such that $0 < z < h$ -- let us consider the infinite series of points:

$$\left.\begin{array}{c} A_1 \equiv (x, y, -z),\ A_2 \equiv (x, y, 2h + z),\ A_3 \equiv (x, y, -2h - z), \ldots, \\ A_{2n} \equiv (x, y, 2nh + z),\ A_{2n+1} \equiv (x, y, -2nh - z), \ldots \\ A'_1 \equiv (x, y, 2h - z),\ A'_2 \equiv (x, y, -2h + z),\ A'_3 \equiv (x, y, 4h - z), \ldots, \\ A'_{2n-1} \equiv (x, y, 2nh - z),\ A'_{2n} \equiv (x, y, -2nh + z), \ldots \end{array}\right|$$

which are the two series of successive images of point A with respect to the two planes σ_1 and σ_2, the images being all located outside of S, except A_1, which may fall with A on plane σ_1, and A'_1, which may fall with A on plane σ_2. Let us call /14

$$r_1,\ r_2,\ r_3, \ldots,\ r_{2n},\ \ r_{2n+1}, \ldots$$

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* Numbers in the margin indicate pagination in the original foreign text.
** In this second report on elastic equilibrium equations for an isotropic body, just as in those which may still follow it, we will continue to apply the prinicples established in the first report. Every time it becomes necessary to refer to the results established in that report, we will add the indicator (I) to the citation.

$$r'_1, r'_2, r'_3, \ldots, r'_{2n-1}, r'_{2n}, \ldots$$

the relative distances of each of these points from the same point (ξ, η, ζ) in S, and let us continue to call r the distance from this point (ξ, η, ζ) to A. When the point (ξ, η, ζ) falls on σ_1, we will have

$$r = r_1,\ r_2 = r_3, \ldots, r_{2n} = r_{2n+1}, \ldots,$$
$$r'_1 = r'_2,\ r'_3 = r'_4, \ldots, r'_{2n-1} = r'_{2n}, \ldots,$$

wherever A may be in S, while when point (ξ, η, ζ) falls on σ_2 we will have

$$r_1 = r_2,\ r_3 = r_4, \ldots, r_{2n-1} = r_{2n}, \ldots,$$
$$r = r'_1,\ r'_2 = r'_3, \ldots, r'_{2n} = r'_{2n+1}, \ldots$$

Thus when point (ξ, η, ζ) is fixed somewhere in S, we have

$$r = r_1,\ r_2 = r'_1,\ r_3 = r'_2, \ldots, r_n = r'_{n-1}, \ldots,$$

when A is chosen on σ_1, while when A is chosen on σ_2 we have

$$r = r'_1,\ r'_2 = r_1,\ r'_3 = r_2, \ldots, r'_n = r_{n-1}, \ldots$$

In the space region $\zeta \geqq 0$ the function $\frac{1}{r_{2n}} - \frac{1}{r_{2n+1}}$, assumed to be dependent on ξ, η, ζ, is harmonic if it becomes zero at infinity on plane σ_1, while it becomes infinitely large and positive when point (ξ, η, ζ) falls in A_{2n}, which belongs to this portion of space. From this we obtain in the region under consideration, $\zeta \geqq 0$, and hence also in S

$$\frac{1}{r_{2n}} - \frac{1}{r_{2n+1}} \geqq 0,$$

for which the series

$$g = \left(\frac{1}{r_2} - \frac{1}{r_3}\right) + \left(\frac{1}{r_4} - \frac{1}{r_5}\right) + \cdots = \sum_1^\infty {}_n \left(\frac{1}{r_{2n}} - \frac{1}{r_{2n+1}}\right), \tag{1}$$

assumed to be dependent on ξ, η, ζ, is a series of harmonic functions, regular in S and all positive, except on plane σ_1 and at infinity where they become zero. On σ_1, series g is identically zero, while on σ_2 it is reduced to $\frac{1}{r_2} - \frac{1}{r_1}$. Thus, based on the theorems of Volterra and of Harnack, g is a series /15 converging absolutely and to an equal degree in S and representing a harmonic and regular function in this region and also in a wider region. In the same field, g is differentiable term by term any number of times, and the derivative series are also harmonic and regular functions.

The series

$$g' = \left(\frac{1}{r'_2} - \frac{1}{r'_3}\right) + \left(\frac{1}{r'_4} - \frac{1}{r'_5}\right) + \cdots = \sum_1^\infty {}_n \left(\frac{1}{r'_{2n}} - \frac{1}{r'_{2n+1}}\right) \tag{2}$$

have similar properties. This becomes zero identically on σ_2, while it becomes equal to $\frac{1}{r'_1}$ on σ_1.

Green's function G relative to space S and to point A inside S is therefore

$$G=\frac{1}{r}-\frac{1}{r_1}-\frac{1}{r'_1}+g+g'=\frac{1}{r}-\frac{1}{r_1}-\frac{1}{r'_1}+ \\ +\sum_1^\infty{}_n\left(\frac{1}{r_{2n}}-\frac{1}{r_{2n+1}}\right)+\sum_1^\infty{}_n\left(\frac{1}{r'_{2n}}-\frac{1}{r'_{2n+1}}\right). \tag{3}$$

For the normal derivative of G, we will have on σ_1

$$\left(\frac{\partial G}{\partial \zeta}\right)_{\zeta=0}=2\left[\frac{z}{r^3}+\sum_1^\infty{}_n\left(\frac{2nh+z}{r^3_{2n}}-\frac{2nh-z}{r'^3_{2n-1}}\right)\right]_{\zeta=0}= \\ =-2\left[\frac{\partial\frac{1}{r}}{\partial z}+\sum_1^\infty{}_n\left(\frac{\partial\frac{1}{r_{2n}}}{\partial z}+\frac{\partial\frac{1}{r'_{2n-1}}}{\partial z}\right)\right]_{\zeta=0}, \tag{4}$$

and on σ_2

$$-\left(\frac{\partial G}{\partial \zeta}\right)_{\zeta=h}=2\left[\frac{h-z}{r^3}+\sum_1^\infty{}_n\left(\frac{(2n+1)h-z}{r'^3_{2n}}-\frac{(2n-1)h+z}{r^3_{2n-1}}\right)\right]_{\zeta=h}= \\ =2\left[\frac{\partial\frac{1}{r}}{\partial z}+\sum_1^\infty{}_n\left(\frac{\partial\frac{1}{r'_{2n}}}{\partial z}+\frac{\partial\frac{1}{r_{2n-1}}}{\partial z}\right)\right]_{\zeta=h}. \tag{4'}$$

Therefore, if Φ is a harmonic function regular in S and becomes zero in points at infinity in S with a higher order than $\frac{1}{r}$, and point A is inside S, the value of function Φ at point A may be represented by formula /16

$$2\pi\Phi=\int_{\sigma_1}\Phi\left[\frac{z}{r^3}+\sum_1^\infty{}_n\left(\frac{2nh+z}{r^3_{2n}}-\frac{2nh-z}{r'^3_{2n-1}}\right)\right]d\sigma+ \\ +\int_{\sigma_2}\Phi\left[\frac{h-z}{r^3}+\sum_1^\infty{}_n\left(\frac{(2n+1)h-z}{r'^3_{2n}}-\frac{(2n-1)h+z}{r^3_{2n-1}}\right)\right]d\sigma= \\ =-\int_{\sigma_1}\Phi\left[\frac{\partial\frac{1}{r}}{\partial z}+\sum_1^\infty{}_n\left(\frac{\partial\frac{1}{r_{2n}}}{\partial z}+\frac{\partial\frac{1}{r'_{2n-1}}}{\partial z}\right)\right]d\sigma+ \\ +\int_{\sigma_2}\Phi\left[\frac{\partial\frac{1}{r}}{\partial z}+\sum_1^\infty{}_n\left(\frac{\partial\frac{1}{r'_{2n}}}{\partial z}+\frac{\partial\frac{1}{r_{2n-1}}}{\partial z}\right)\right]d\sigma. \tag{5}$$

Noting that -- since there are two series of harmonic and regular functions in S

$$\sum_1^\infty{}_n\int_{\sigma_1}\Phi\left(\frac{2nh+z}{r^3_{2n}}-\frac{2nh-z}{r'^3_{2n-1}}\right)d\sigma,$$

$$\sum_1^\infty{}_n\int_{\sigma_2}\Phi\left(\frac{(2n+1)h-z}{r'^3_{2n}}-\frac{(2n-1)h+z}{r^3_{2n-1}}\right)d\sigma,$$

converging to an equal degree on σ_1 and σ_2, and therefore in S -- term by term integration is permissible in expression (5), we may also write

$$\left.\begin{aligned}2\pi\Phi &= \int_{\sigma_1}\Phi\frac{z}{r^3}d\sigma + \sum_1^\infty{}^n\int_{\sigma_1}\Phi\left(\frac{2nh+z}{r_{2n}^3} - \frac{2nh-z}{r'^3_{2n-1}}\right)d\sigma + \\ &+ \int_{\sigma_2}\Phi\frac{h-z}{r^3}d\sigma + \sum_1^\infty{}^n\int_{\sigma_2}\Phi\left(\frac{(2n+1)h-z}{r'^3_{2n}} - \frac{(2n-1)h+z}{r_{2n-1}^3}\right)d\sigma = \\ &= -\frac{\partial}{\partial z}\int_{\sigma_1}\frac{\Phi}{r}d\sigma - \sum_1^\infty{}^n\left(\frac{\partial}{\partial z}\int_{\sigma_1}\frac{\Phi}{r_{2n}}d\sigma + \frac{\partial}{\partial z}\int_{\sigma_1}\frac{\Phi}{r'_{2n-1}}d\sigma\right) + \\ &+ \frac{\partial}{\partial z}\int_{\sigma_2}\frac{\Phi}{r}d\sigma + \sum_1^\infty{}^n\left(\frac{\partial}{\partial z}\int_{\sigma_2}\frac{\Phi}{r'_{2n}}d\sigma + \frac{\partial}{\partial z}\int_{\sigma_2}\frac{\Phi}{r_{2n-1}}d\sigma\right).\end{aligned}\right\}\quad(5')$$

It is conversely also easy to demonstrate that whatever are the values assigned to Φ on σ_1 and σ_2, provided that they form finite and continuous functions of the points in these two planes and are such that $\rho\ \Phi$, $\rho = \sqrt{(x-\xi)^2 + (y-\eta)^2}$, becomes zero in points at infinity in σ_1 and σ_2, expression (5) or (5') defines a function of point A which is harmonic and regular inside S and tends toward the assigned values in the points in σ. And in fact if Φ satisfies the preceding assumptions, each of the terms in expression (5') is finite, wherever A may be in S, and represents a harmonious and regular function inside S. The two series /17

$$\sum_1^\infty{}^n\int_{\sigma_1}\Phi\left(\frac{2nh+z}{r_{2n}^3} - \frac{2nh-z}{r'^3_{2n-1}}\right)d\sigma,$$

$$\sum_1^\infty{}^n\int_{\sigma_2}\Phi\left(\frac{(2n+1)h-z}{r'^3_{2n}} - \frac{(2n-1)h+z}{r_{2n-1}^3}\right)d\sigma$$

converge in equal degree on σ_1 and σ_2; hence they and function Φ defined by expression (5') are harmonic and regular functions inside S. If it is observed that

$$\lim_{z=0}\sum_1^\infty{}^n\int_{\sigma_1}\Phi\left(\frac{2nh+z}{r_{2n}^3} - \frac{2nh-z}{r'^3_{2n-1}}\right)d\sigma = 0$$

$$\lim_{z=0}\left\{\int_{\sigma_2}\Phi\frac{h-z}{r^3}d\sigma + \sum_1^\infty{}^n\int_{\sigma_2}\Phi\left(\frac{(2n+1)h-z}{r'^3_{2n}} - \frac{(2n-1)h+z}{r_{2n-1}^3}\right)d\sigma\right\} = 0,$$

we have

$$\lim_{z=0}\Phi = \frac{1}{2\pi}\lim_{z=0}\int_{\sigma_1}\Phi\frac{z}{r^3}d\sigma$$

and therefore Φ in the points in σ_1 tends toward the assigned values. The same may be said for the points in σ_2. In short, the two series

$$\sum_1^\infty{}^n\left(\frac{2nh+z}{r_{2n}^3} - \frac{2nh-z}{r'^3_{2n-1}}\right)_{\zeta=0},$$

$$\sum_1^\infty {}_n \left(\frac{(2n+1)h - z}{r'^3_{2n}} - \frac{(2n-1)h + z}{r^3_{2n-1}}\right)_{\zeta=h},$$

considered as dependent on x, y, z are convergent to an equal degree on σ_1 and σ_2, and therefore also inasmuch as their terms from a certain moment onward (when A, always included in S, is sufficiently distant) are positive and less than the terms of the respective series

$$2z \sum_1^\infty {}_n \frac{1}{r^3_{2n}}, \qquad 2(h-z) \sum_1^\infty {}_n \frac{1}{r'^3_{2n}}$$

and since these converge and become zero in points at infinity in σ_1 and in σ_2, at least like $\frac{1}{\rho}$, the integrals which appear in expression (5) are also /18
finite and represent harmonic and regular functions in S. Since term by term integration is permitted in expression (5), Φ may also be represented by expression (5).

The analytical expressions obtained for Φ could also have been derived by applying Schwarz's alternate procedure. The terms containing integrals extended over σ_1 are in fact the successive reflections on planes σ_1 and σ_2 of the harmonic and regular function in region $\zeta \geqq 0$, which in σ_1 assumes the values assigned to Φ in this plane. And the same may be said of the terms which contain integrals extended over σ_2.

2. As a consequence of the absolute and uniform convergence of series g in S the absolute and uniform convergence of series

$$\begin{aligned} \bar{g} &= \left(\frac{1}{r_2} - \frac{1}{r_3}\right) - \left(\frac{1}{r_4} - \frac{1}{r_5}\right) + \left(\frac{1}{r_6} - \frac{1}{r_7}\right) - \cdots = \\ &= -\sum_1^\infty {}_n (-1)^n \left(\frac{1}{r_{2n}} - \frac{1}{r_{2n+1}}\right). \end{aligned} \tag{6}$$

is likewise deduced. $\bar{g}$, considered in S as dependent on ξ, η, ζ, possesses properties similar to those of g. It becomes zero identically on σ_1, while on σ_2 it is reduced to

$$\frac{1}{r_2} - \left(\frac{2}{r_3} - \frac{2}{r_5}\right) - \left(\frac{2}{r_7} - \frac{2}{r_9}\right) - \cdots = \frac{1}{r_2} - 2\sum_0^\infty {}_n \left(\frac{1}{r_{4n+3}} - \frac{1}{r_{4n+5}}\right).$$

Instead, the normal derivative of $\bar{g}$ on σ_1 becomes

$$\left(\frac{\partial \bar{g}}{\partial \zeta}\right)_{\zeta=0} = -2\sum_1^\infty {}_n (-1)^n \frac{2nh+z}{r^3_{2n}},$$

while on σ_2 it is reduced to $-\left(\frac{\partial \frac{1}{r_1}}{\partial \zeta}\right)_{\zeta=h}$. Similarly, from the convergence of series

$$g' = \frac{1}{r'_2} + \left(\frac{1}{r'_4} - \frac{1}{r'_3}\right) + \left(\frac{1}{r'_6} - \frac{1}{r'_5}\right) + \cdots = \frac{1}{r'_2} + \sum_{2}^{\infty}{}_n \left(\frac{1}{r'_{2n}} - \frac{1}{r'_{2n-1}}\right)$$

whose convergence is absolute and of equal degree in S, the absolute and equal-degree convergence of series

$$\bar{g}' = \frac{1}{r'_2} - \sum_{2}^{\infty}{}_n (-1)^n \left(\frac{1}{r'_{2n}} - \frac{1}{r'_{2n-1}}\right) \tag{7}$$

is derived. This series possesses properties like those of g' in S -- on σ_1 /19
it reduces to $\frac{1}{r'2}$, while on σ_2 it reduces to

$$\left(\frac{2}{r'_2} - \frac{2}{r'_4}\right) + \left(\frac{2}{r'_6} - \frac{2}{r'_8}\right) + \cdots = 2 \sum_{0}^{\infty}{}_n \left(\frac{1}{r'_{4n+2}} - \frac{1}{r'_{4(n+1)}}\right).$$

Instead, for the normal derivative $\bar{g}'$ we find that on σ_1 it reduces to

$$\left(\frac{\partial \bar{g}'}{\partial \zeta}\right)_{\zeta=0} = \frac{-2h + z}{r'^3_2} - 2 \sum_{2}^{\infty}{}_n (-1)^n \frac{-2nh + z}{r'^3_{2n-1}},$$

while on σ_2 it is identically zero. Hence, function

$$\bar{G} = \frac{1}{r} - \frac{1}{r_1} + \frac{1}{r'_1} - \bar{g} - \bar{g}' \tag{8}$$

becomes zero identically on σ_1, while on σ_2 it is the normal derivative which becomes zero identically. G is therefore the Green function which solves the problem of determining the function which is harmonic and regular in S and such that it assumes given values on σ_1, while its normal derivative assumes given values on σ_2. Its analytic expression will be given by

$$\left.\begin{aligned}
2\pi\bar{\Phi} &= \int_{\sigma_1} \bar{\varphi} \left[\frac{z}{r^3} + \sum_{1}^{\infty}{}_n (-1)^n \frac{2nh + z}{r^3_{2n}} + \sum_{1}^{\infty}{}_n (-1)^n \frac{-2nh + z}{r'^3_{2n-1}}\right] d\sigma + \\
&+ \int_{\sigma_2} \frac{\partial \bar{\Phi}}{\partial \zeta} \left[\frac{1}{r} - \frac{1}{r_1} + \sum_{0}^{\infty}{}_n \left(\frac{1}{r_{4n+3}} - \frac{1}{r_{4n+5}}\right) - \sum_{0}^{\infty}{}_n \left(\frac{1}{r'_{4n+2}} - \frac{1}{r'_{4(n+1)}}\right)\right] d\sigma = \\
&= -\frac{\partial}{\partial z} \int_{\sigma_1} \bar{\Phi} \frac{d\sigma}{r} - \sum_{1}^{\infty}{}_n (-1)^n \frac{\partial}{\partial z} \int_{\sigma_1} \bar{\Phi} \frac{d\sigma}{r_{2n}} - \sum_{1}^{\infty}{}_n (-1)^n \frac{\partial}{\partial z} \int_{\sigma_1} \bar{\Phi} \frac{d\sigma}{r'_{2n-1}} + \\
&\quad + \int_{\sigma_2} \frac{\partial \bar{\Phi}}{\partial \zeta} \frac{d\sigma}{r} - \int_{\sigma_2} \frac{\partial \bar{\Phi}}{\partial \zeta} \frac{d\sigma}{r_1} + \sum_{0}^{\infty}{}_n \left(\int_{\sigma_2} \frac{\partial \bar{\Phi}}{\partial \zeta} \frac{d\sigma}{r_{4n+3}} - \int_{\sigma_2} \frac{\partial \bar{\Phi}}{\partial \zeta} \frac{d\sigma}{r_{4n+5}} - \right. \\
&\quad \left. - \int_{\sigma_2} \frac{\partial \Phi}{\partial \zeta} \frac{d\sigma}{r'_{4n+2}} + \int_{\sigma_2} \frac{\partial \Phi}{\partial \zeta} \frac{d\sigma}{r'_{4(n+1)}}\right).
\end{aligned}\right\} \tag{9}$$

3. To determine the analytic expression of the harmonic function whose normal derivative assumes given values on σ_1 and σ_2, let us start from formula (5), which we will be able to write as /20

$$2\pi\Phi = -\frac{\partial}{\partial z}\int_{\sigma_1}\Phi\frac{d\sigma}{r} - \int_{\sigma_1}\Phi\left(\sum_1^{\infty}{}_n\frac{\partial}{\partial z}\frac{1}{r_{2n}}\right)d\sigma - \int_{\sigma_1}\Phi\left(\sum_1^{\infty}{}_n\frac{\partial}{\partial z}\frac{1}{r'_{2n-1}}\right)d\sigma +$$
$$+\frac{\partial}{\partial z}\int_{\sigma_2}\Phi\frac{d\sigma}{r} + \int_{\sigma_2}\Phi\left(\sum_1^{\infty}{}_n\frac{\partial}{\partial z}\frac{1}{r'_{2n}}\right)d\sigma + \int_{\sigma_2}\Phi\left(\sum_1^{\infty}{}_n\frac{\partial}{\partial z}\frac{1}{r_{2n-1}}\right)d\sigma.$$

In the series which appear under the integrals, it is not permissible to reverse the summation sign and the derivative sign, because the series $\sum_1^{\infty} n\,\frac{1}{r2n}$, ... are divergent. Let us note, however, that for the Mittag-Leffler theorem extended to the harmonic functions of Professor Appell*, the series

$$\sum_1^{\infty}{}_n\left(\frac{1}{r_{2n}}-\frac{1}{2nh}\right),\quad \sum_1^{\infty}{}_n\left(\frac{1}{r'_{2n-1}}-\frac{1}{2nh}\right),$$
$$\sum_1^{\infty}{}_n\left(\frac{1}{r'_{2n}}-\frac{1}{2nh}\right),\quad \sum_1^{\infty}{}_n\left(\frac{1}{r_{2n-1}}-\frac{1}{2nh}\right)$$

are absolutely convergent and in equal degree, except in the points in which some r become zero, for which -- noting that in space S none of the r's which appear in this series become zero -- we may write

$$\sum_1^{\infty}{}_n\frac{\partial}{\partial z}\frac{1}{r_{2n}} = \sum_1^{\infty}{}_n\frac{\partial}{\partial z}\left(\frac{1}{r_{2n}}-\frac{1}{2nh}\right) = \frac{\partial}{\partial z}\sum_1^{\infty}{}_n\left(\frac{1}{r_{2n}}-\frac{1}{2nh}\right),$$

and therefore

$$\left.\begin{aligned}2\pi\Phi = \frac{\partial}{\partial z}\Big\{&-\int_{\sigma_1}\Phi\frac{d\sigma}{r} - \sum_1^{\infty}{}_n\int_{\sigma_1}\Phi\left(\frac{1}{r_{2n}}-\frac{1}{2nh}\right)d\sigma - \\ &-\sum_1^{\infty}{}_n\int_{\sigma_1}\Phi\left(\frac{1}{r'_{2n-1}}-\frac{1}{2nh}\right)d\sigma + \int_{\sigma_2}\Phi\frac{d\sigma}{r} + \sum_1^{\infty}{}_n\int_{\sigma_2}\Phi\left(\frac{1}{r'_{2n}}-\frac{1}{2nh}\right)d\sigma + \\ &+\sum_1^{\infty}{}_n\int_{\sigma_2}\Phi\left(\frac{1}{r_{2n-1}}-\frac{1}{2nh}\right)d\sigma\Big\}.\end{aligned}\right\} \quad (5'')$$

From this formula it may be deduced that the function which is harmonic and regular in S /21

$$\left.-\int_{\sigma_1}\Phi\frac{d\sigma}{r} - \sum_1^{\infty}{}_n\int_{\sigma_1}\Phi\left(\frac{1}{r_{2n}}-\frac{1}{2nh}\right)d\sigma - \sum_1^{\infty}{}_n\int_{\sigma_1}\Phi\left(\frac{1}{r'_{2n-1}}-\frac{1}{2nh}\right)d\sigma - \right\} \quad (10)$$

* "On the Functions of Three Real Variables...," Acta Matematica, Vol. 4, p. 326.

$$-\int_{\sigma_2} \Phi \frac{d\sigma}{r} - \sum_1^\infty{}_n \int_{\sigma_2} \Phi\left(\frac{1}{r'_{2n}} - \frac{1}{2nh}\right) d\sigma - \sum_1^\infty{}_n \int_{\sigma_2} \Phi\left(\frac{1}{r_{2n-1}} - \frac{1}{2nh}\right) d\sigma \Bigg| \Bigg\} \qquad (10)$$

is such that its normal derivative acquires the given values of $2\pi\Phi$ in σ_1 and σ_2.

Expression (10) also shows that a function harmonic and regular in S may always be put in the form of a derivative with respect to z of a function also harmonic and regular in S.

4. Let us now add the following considerations, which we will use constantly in the following. Let us recall that any function which is harmonic and regular in the region z > 0 is representable by means of a series of the form

$$\sum_m \int_0^\infty e^{-\gamma z} \left[a_m(\gamma) \cos m\,\omega + b_m(\gamma) \operatorname{sen}^{**} m\,\omega\right] J_m(\gamma\rho)\, d\gamma \,(^*) \qquad (11)$$

which is absolutely and uniformly convergent in that region, where ρ, ω, z is a system of cylindrical coordinates; J_m, the Bessel function of the first kind and of order m; $a_m(\gamma)$, $b_m(\gamma)$, two functions of γ; and m, a positive whole number which can vary from zero to $+\infty$. Conversely, any series like expression (11) represents a harmonic and regular function in the region z > 0 which becomes zero for $z = +\infty$. We have assumed that the region is the one determined by inequality z > 0, but it is clear that the same holds true in any region limited by a plane.

More generally, any function which is harmonic and regular in region S, i.e., 0 < z < h, is representable by means of an expression of form

$$\left.\begin{aligned} &\sum_m \int_0^\infty e^{-\gamma z} \left[a_m(\gamma) \cos m\,\omega + b_m(\gamma) \operatorname{sen} m\,\omega\right] J_m(\gamma\rho)\, d\gamma\, + \\ &+ \sum_m \int_0^\infty e^{-\gamma(h-z)} \left[\bar{a}_m(\gamma) \cos m\,\omega + \bar{b}_m(\gamma) \operatorname{sen} m\,\omega\right] J_m(\gamma\rho)\, d\gamma \end{aligned}\right\} \qquad (12)$$

the two series being absolutely and uniformly convergent in this field, and vice versa, as above.

This last result may be easily deduced from expression (5) by developing the individual terms in accord with formula (11) and summing with respect to n. /22

Finally, let us observe that, if an expression like expression (12) is

* Heine, Handb. der Kugelf. (Manual of Spherical Functions), Vol. 2, p. 189.
** Sen is correctly sin in English terminology.

identically zero in S, the functions $a_m(\gamma)$, $b_m(\gamma)$; $\bar{a}_m(\gamma)$, $\bar{b}_m(\gamma)$ must be zero.

5. *Case in which u, v, w are given on the two limiting planes.* Let us assume that the values of u, v, w given on σ are such by nature that the results of the preceding sections and formulas (5) or (5') and similar formulas from Report I may be applied. If we then designate by U, V, W the functions harmonic and regular in S which on σ acquire, respectively, the given values u, v, w, for the formulas just cited we will have

$$\left.\begin{aligned}
u = {} & U + \frac{\lambda+\mu}{4\pi\mu}\Big\{ z\frac{\partial}{\partial x}\int_{\sigma_1}\frac{\theta}{r}\,d\sigma + \sum_1^\infty{}_n (2nh+z)\frac{\partial}{\partial x}\int_{\sigma_1}\frac{\theta}{r_{2n}}\,d\sigma - \\
& - \sum_1^\infty{}_n (2nh-z)\frac{\partial}{\partial x}\int_{\sigma_1}\frac{\theta}{r'_{2n-1}}\,d\sigma + (h-z)\frac{\partial}{\partial x}\int_{\sigma_2}\frac{\theta}{r}\,d\sigma + \\
& + \sum_1^\infty{}_n [(2n+1)h-z]\frac{\partial}{\partial x}\int_{\sigma_2}\frac{\theta}{r'_{2n}}\,d\sigma - \\
& - \sum_1^\infty{}_n [(2n-1)h+z]\frac{\partial}{\partial x}\int_{\sigma_2}\frac{\theta}{r_{2n-1}}\,d\sigma\Big\}, \\
v = {} & V + \frac{\lambda+\mu}{4\pi\mu}\Big\{ z\frac{\partial}{\partial y}\int_{\sigma_1}\frac{\theta}{r}\,d\sigma + \sum_1^\infty{}_n (2nh+z)\frac{\partial}{\partial y}\int_{\sigma_1}\frac{\theta}{r_{2n}}\,d\sigma - \\
& - \sum_1^\infty{}_n (2nh-z)\frac{\partial}{\partial y}\int_{\sigma_1}\frac{\theta}{r'_{2n-1}}\,d\sigma + (h-z)\frac{\partial}{\partial y}\int_{\sigma_2}\frac{\theta}{r}\,d\sigma + \\
& + \sum_1^\infty{}_n [(2n+1)h-z]\frac{\partial}{\partial y}\int_{\sigma_2}\frac{\theta}{r'_{2n}}\,d\sigma - \\
& - \sum_1^\infty{}_n [(2n-1)h+z]\frac{\partial}{\partial y}\int_{\sigma_2}\frac{\theta}{r_{2n-1}}\,d\sigma\Big\}, \\
w = {} & W - \frac{\lambda+\mu}{2\mu} z\theta + \frac{\lambda+\mu}{4\pi\mu} h\Big[\frac{\partial}{\partial z}\int_{\sigma_2}\frac{\theta}{r}\,d\sigma + \\
& + \sum_1^\infty{}_n \frac{\partial}{\partial z}\int_{\sigma_2}\frac{\theta}{r'_{2n}}\,d\sigma + \sum_1^\infty{}_n \frac{\partial}{\partial z}\int_{\sigma_2}\frac{\theta}{r_{2n-1}}\,d\sigma\Big].
\end{aligned}\right\} \tag{13}$$

And, if we set /23

$$\left.\begin{aligned}
\varphi_0 &= \int_{\sigma_1}\frac{\theta}{r}\,d\sigma, & \varphi_n &= \int_{\sigma_1}\frac{\theta}{r_{2n}}\,d\sigma, & \varphi_{-n} &= \int_{\sigma_1}\frac{\theta}{r'_{2n-1}}\,d\sigma; \\
\varphi'_0 &= \int_{\sigma_2}\frac{\theta}{r}\,d\sigma, & \varphi'_n &= \int_{\sigma_2}\frac{\theta}{r'_{2n}}\,d\sigma, & \varphi'_{-n} &= \int_{\sigma_2}\frac{\theta}{r_{2n-1}}\,d\sigma,
\end{aligned}\right\} \tag{14}$$

we will also be able to write more briefly

$$u = U + \frac{\lambda+\mu}{4\pi\mu}\sum_{-\infty}^{+\infty}{}_n\Big[(2nh+z)\frac{\partial\varphi_n}{\partial x} + (2nh+h-z)\frac{\partial\varphi'_n}{\partial x}\Big], \Big\} \tag{13'}$$

$$
\left.\begin{aligned}
v &= V + \frac{\lambda+\mu}{4\pi\mu}\sum_{-\infty}^{+\infty}{}^{n}\left[(2nh+z)\frac{\partial\varphi_n}{\partial y} + (2nh+h-z)\frac{\partial\varphi'_n}{\partial y}\right],\\
w &= W - \frac{\lambda+\mu}{2\mu}z\theta + \frac{\lambda+\mu}{4\pi\mu}h\sum_{-\infty}^{+\infty}{}^{n}\frac{\partial\varphi'_n}{\partial z},\\
\theta &= \frac{1}{2\pi}\sum_{-\infty}^{+\infty}{}^{n}\frac{\partial}{\partial z}(\varphi'_n-\varphi_n).
\end{aligned}\right\} \tag{13'}
$$

The problem which we have proposed will apparently be solved if we succeed in determining functions ϕ so that: (1) equation

$$
\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{15}
$$

is identically satisfied in S; (2) in the same region S and on σ the values ϕ and their first derivatives are finite; (3) the values of ϕ themselves are harmonic and regular in S; (4) for z = 0:

$$
\varphi'_0=\varphi'_{-1},\quad \varphi_1=\varphi_{-1},\quad \varphi'_{-2}=\varphi'_1,\ldots,\quad \varphi_n=\varphi_{-n},\quad \varphi'_{-n}=\varphi'_{n-1},\ldots
$$

while for z = h:

$$
\varphi_0=\varphi_1,\quad \varphi'_1=\varphi'_{-1},\quad \varphi_{-2}=\varphi_1,\ldots,\quad \varphi'_n=\varphi'_{-n},\quad \varphi_{-n}=\varphi_{n-1},\ldots;
$$

(5) finally, the series which appear on the right sides of equations (13) must be convergent and differentiable at least twice in S.

Therefore let us assume that /24

$$
\left.\begin{aligned}
\varphi_0 &= \sum{}^{m}\int_0^\infty e^{-\gamma z}(a_m\cos m\omega + b_m \operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma,\\
\varphi'_0 &= \sum{}^{m}\int_0^\infty e^{-\gamma(h-z)}(\bar a_m\cos m\omega + \bar b_m \operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma
\end{aligned}\right\} \tag{16}
$$

in which a_m, b_m, $\bar a_m$, $\bar b_m$ are functions of γ to be determined. From this we consequently have

$$
\left.\begin{aligned}
\varphi_n &= \sum{}^{m}\int_0^\infty e^{-\gamma(2nh+z)}(a_m\cos m\omega + b_m \operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma,\\
\varphi_{-n} &= \sum{}^{m}\int_0^\infty e^{-\gamma(2nh-z)}(a_m\cos m\omega + b_m \operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma,\\
\varphi'_n &= \sum{}^{m}\int_0^\infty e^{-\gamma(2nh+h-z)}(\bar a_m\cos m\omega + \bar b_m \operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma,\\
\varphi'_{-n} &= \sum{}^{m}\int_0^\infty e^{-\gamma(2nh-h+z)}(\bar a_m\cos m\omega + \bar b_m \operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma.
\end{aligned}\right\} \tag{16'}
$$

With these values for ϕ, an attempt must now be made to satisfy equation (15), which may be written

$$
\left.\begin{aligned}
&\frac{\lambda+3\mu}{4\pi\mu}\sum_{-\infty}^{+\infty}{}_n \frac{\partial}{\partial z}(\varphi'_n-\varphi_n)+\\
&+\frac{\lambda+\mu}{2\pi\mu}h\sum_{-\infty}^{+\infty}{}_n n\frac{\partial^2}{\partial z^2}(\varphi_n+\varphi'_n)=\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}.
\end{aligned}\right\} \tag{15'}
$$

The second term in this equation, since it is a harmonic and regular function S, may in this region be represented by an expression of the form

$$
\left.\begin{aligned}
&\sum{}_m\int_0^\infty e^{-\gamma z}(a_m\cos m\omega+b_m\operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma+\\
&+\sum{}_m\int_0^\infty e^{-\gamma(h-z)}(\bar a_m\cos m\omega+\bar b_m\operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma
\end{aligned}\right\} \tag{17}
$$

where a_m, b_m, $\bar{a}_m$, $\bar{b}_m$ are functions of γ which are assumed to be known. Substituting into expression (15') the values of ϕ, ϕ' and of $\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}$ given by expressions (16), (16'), and (17) and setting the coefficients of

$$
e^{-\gamma z}\cos m\omega\, J_m(\gamma\rho),\quad e^{-\gamma z}\operatorname{sen} m\omega\, J_m(\gamma\rho),\quad e^{\gamma(h-z)}\cos m\omega\, J_m(\gamma\rho),\quad e^{-\gamma(h-z)}\operatorname{sen} m\omega\, J_m(\gamma\rho)
$$

in the two terms equal, the following equations are obtained /25

$$
\left.\begin{aligned}
a_m&=\frac{\lambda+3\mu}{4\pi\mu}\gamma\left[a_m\left(1+\sum_1^\infty{}_n e^{-2nh\gamma}\right)-\bar a_m\sum_1^\infty{}_n e^{-(2n-1)h\gamma}\right]+\\
&+\frac{\lambda+\mu}{2\pi\mu}h\gamma^2\left[a_m\sum_1^\infty{}_n n e^{-2nh\gamma}-\bar a_m\sum_1^\infty{}_n n e^{-(2n-1)h\gamma}\right],\\
b_m&=\frac{\lambda+3\mu}{4\pi\mu}\gamma\left[b_m\left(1+\sum_1^\infty{}_n e^{-2nh\gamma}\right)-\bar b_m\sum_1^\infty{}_n e^{-(2n-1)h\gamma}\right]+\\
&+\frac{\lambda+\mu}{2\pi\mu}h\gamma^2\left[b_m\sum_1^\infty{}_n n e^{-2nh\gamma}-\bar b_m\sum_1^\infty{}_n n e^{-(2n-1)h\gamma}\right],\\
\bar a_m&=\frac{\lambda+3\mu}{4\pi\mu}\gamma\left[\bar a_m\left(1+\sum_1^\infty{}_n e^{-2nh\gamma}\right)-a_m\sum_1^\infty{}_n e^{-(2n-1)h\gamma}\right]+\\
&+\frac{\lambda+\mu}{2\pi\mu}h\gamma^2\left[\bar a_m\sum_1^\infty{}_n n e^{-2nh\gamma}-a_m\sum_1^\infty{}_n n e^{-(2n-1)h\gamma}\right],\\
\bar b_m&=\frac{\lambda+3\mu}{4\pi\mu}\gamma\left[\bar b_m\left(1+\sum_1^\infty{}_n e^{-2nh\gamma}\right)-b_m\sum_1^\infty{}_n e^{-(2n-1)h\gamma}\right]+\\
&+\frac{\lambda+\mu}{2\pi\mu}h\gamma^2\left[\bar b_m\sum_1^\infty{}_n n e^{2nh\gamma}-b_m\sum_1^\infty{}_n n e^{-(2n-1)h\gamma}\right],
\end{aligned}\right\} \tag{18}
$$

and

$$
\left.\begin{aligned}
&\frac{4\pi\mu}{\gamma e^{h\gamma}}(1-e^{2h\gamma})^2 a_m=(\lambda+3\mu)(e^{2h\gamma}-1)(a_m e^{h\gamma}-\bar a_m)+\\
&\qquad+2(\lambda+\mu)h\gamma e^{h\gamma}(a_m-\bar a_m e^{h\gamma}),\\
&-\frac{4\pi\mu}{\gamma e^{h\gamma}}(1-e^{2h\gamma})^2\bar a_m=(\lambda+3\mu)(e^{2h\gamma}-1)(a_m-\bar a_m e^{h\gamma})+
\end{aligned}\right\} \tag{18'}
$$

$$+ 2(\lambda + \mu) h \gamma e^{h\gamma} (a_m e^{h\gamma} - \bar{a}_m), \tag{18'}$$

. . . .

$$\left.\begin{aligned}
\Delta a_m &= [(\lambda + 3\mu)(e^{2h\gamma} - 1) + 2(\lambda + \mu) h \gamma] e^{h\gamma} a_m + \\
&\quad + [(\lambda + 3\mu)(e^{2h\gamma} - 1) + 2(\lambda + \mu) h \gamma e^{2h\gamma}] \bar{a}_m, \\
\Delta \bar{a}_m &= [(\lambda + 3\mu)(e^{2h\gamma} - 1) + 2(\lambda + \mu) h \gamma e^{2h\gamma}] a_m + \\
&\quad + [(\lambda + 3\mu)(e^{2h\gamma} - 1) + 2(\lambda + \mu) h \gamma] e^{h\gamma} \bar{a}_m, \\
\Delta b_m &= [(\lambda + 3\mu)(e^{2h\gamma} - 1) + 2(\lambda + \mu) h \gamma] e^{h\gamma} b_m + \\
&\quad + [(\lambda + 3\mu)(e^{2h\gamma} - 1) + 2(\lambda + \mu) h \gamma e^{2h\gamma}] \bar{b}_m, \\
\Delta \bar{b}_m &= [(\lambda + 3\mu)(e^{2h\gamma} - 1) + 2(\lambda + \mu) h \gamma e^{2h\gamma}] b_m + \\
&\quad + [(\lambda + 3\mu)(e^{2h\gamma} - 1) + 2(\lambda + \mu) h \gamma] e^{h\gamma} \bar{b}_m, \\
\Delta &= \frac{\gamma e^{h\gamma}}{4\pi\mu(e^{2h\gamma} - 1)} [(\lambda + 3\mu)^2 (e^{2h\gamma} - 1)^2 - 4(\lambda + \mu)^2 h^2 \gamma^2 e^{2h\gamma}].
\end{aligned}\right\} \tag{19}$$

The values of ϕ_0, ϕ'_0 and therefore also the other values of ϕ compiled /26 according to expressions (16) and (16') with those values of functions a_m, b_m, $\bar{a}_m$, $\bar{b}_m$ are absolutely convergent and of equal degree in region S as a consequence of the convergence of series (17), which is absolute and of equal degree in the same region. The same properties are possessed also by the series $\sum_1^\infty \phi n$, $\sum_1^\infty \phi{-n}$, ... and the others which appear in the second term of expression (13), as is easily verified. If, moreover, U, V, W are finite together with the first and second derivatives also on σ, the ϕ will also possess these properties.

6. With this, the problem is solved. We should like now to add the following developments which serve to complete the analytical representation of the solution. Let us suppose that in S

$$\left.\begin{aligned}
U &= \sum_m \int_0^\infty e^{-\gamma z} (\alpha_m \cos m\omega + \beta_m \operatorname{sen} m\omega) J_m(\gamma\rho)\, d\gamma + \\
&\quad + \sum_m \int_0^\infty e^{-\gamma(h-z)} (\bar{\alpha}_m \cos m\omega + \bar{\beta}_m \operatorname{sen} m\omega) J_m(\gamma\rho)\, d\gamma, \\
V &= \sum_m \int_0^\infty e^{-\gamma z} (\alpha_m^{(1)} \cos m\omega + \beta_m^{(1)} \operatorname{sen} m\omega) J_m(\gamma\rho)\, d\gamma + \\
&\quad + \sum_m \int_0^\infty e^{-\gamma(h-z)} (\bar{\alpha}_m^{(1)} \cos m\omega + \bar{\beta}_m^{(1)} \operatorname{sen} m\omega) J_m(\gamma\rho)\, d\gamma, \\
W &= \sum_m \int_0^\infty e^{-\gamma z} (\alpha_m^{(2)} \cos m\omega + \beta_m^{(2)} \operatorname{sen} m\omega) J_m(\gamma\rho)\, d\gamma + \\
&\quad + \sum_m \int_0^\infty e^{-\gamma(h-z)} (\bar{\alpha}_m^{(2)} \cos m\omega + \bar{\beta}_m^{(2)} \operatorname{sen} m\omega) J_m(\gamma\rho)\, d\gamma,
\end{aligned}\right\} \tag{20}$$

Then, assuming that x = ρ cos ω, y = ρ sin ω: /27

$$\frac{\partial U}{\partial x} = -\frac{1}{2}\cos\omega\int_0^\infty \gamma e^{-\gamma z}\alpha_0 J_1(\gamma\rho)\,d\gamma +$$

$$+\frac{1}{2}\Sigma_m\int_0^\infty \gamma e^{-\gamma z}\left[(\alpha_{m+1}-\alpha_{m-1})\cos m\omega + (\beta_{m+1}-\beta_{m-1})\operatorname{sen} m\omega\right] J_m(\gamma\rho)\,d\gamma -$$

$$-\frac{1}{2}\cos\omega\int_0^\infty \gamma e^{-\gamma(h-z)}\bar{\alpha}_0 J_1(\gamma\rho)\,d\gamma +$$

$$+\frac{1}{2}\Sigma_m\int_0^\infty \gamma e^{-\gamma(h-z)}\left[(\bar{\alpha}_{m+1}-\bar{\alpha}_{m-1})\cos m\omega + (\bar{\beta}_{m+1}-\bar{\beta}_{m-1})\operatorname{sen} m\omega\right] J_m(\gamma\rho)\,d\gamma,$$

$$\frac{\partial V}{\partial y} = -\frac{1}{2}\operatorname{sen}\omega\int_0^\infty \gamma e^{-\gamma z}\alpha_0^{(1)} J_1(\gamma\rho)\,d\gamma +$$

$$+\frac{1}{2}\Sigma_m\int_0^\infty \gamma e^{-\gamma z}\left[-(\alpha^{(1)}_{m+1}+\alpha^{(1)}_{m-1})\operatorname{sen} m\omega + (\beta^{(1)}_{m+1}+\beta^{(1)}_{m-1})\cos m\omega\right] J_m(\gamma\rho)\,d\gamma -$$

$$-\frac{1}{2}\operatorname{sen}\omega\int_0^\infty \gamma e^{-\gamma(h-z)}\bar{\alpha}_0^{(1)} J_1(\gamma\rho)\,d\gamma +$$

$$+\frac{1}{2}\Sigma_m\int_0^\infty \gamma e^{-\gamma(h-z)}\left[-(\bar{\alpha}^{(1)}_{m+1}+\bar{\alpha}^{(1)}_{m-1})\operatorname{sen} m\omega + (\bar{\beta}^{(1)}_{m+1}+\bar{\beta}^{(1)}_{m-1})\cos m\omega\right] J_m(\gamma\rho)\,d\gamma\ (*),$$

$$\frac{\partial W}{\partial z} = -\Sigma_m\int_0^\infty \gamma e^{-\gamma z}(\alpha_m^{(2)}\cos m\omega + \beta_m^{(2)}\operatorname{sen} m\omega) J_m(\gamma\rho)\,d\gamma +$$

$$+\Sigma_m\int_0^\infty \gamma e^{-\gamma(h-z)}(\bar{\alpha}_m^{(2)}\cos m\omega + \bar{\beta}_m^{(2)}\operatorname{sen} m\omega) J_m(\gamma\rho)\,d\gamma,$$

---

* It is sufficient therefore to note that

$$\frac{\partial\rho}{\partial x} = \cos\omega,\quad \frac{\partial\rho}{\partial y} = \operatorname{sen}\omega,\quad \frac{\partial\omega}{\partial x} = -\frac{\operatorname{sen}\omega}{\rho},\quad \frac{\partial\omega}{\partial y} = \frac{\cos\omega}{\rho}$$

$$J'_0(\gamma\rho) = -J_1(\gamma\rho),\quad \frac{2mJ_m(\gamma\rho)}{\gamma\rho} = J_{m-1}(\gamma\rho) + J_{m+1}(\gamma\rho),$$

$$2J'_m(\gamma\rho) = J_{m-1}(\gamma\rho) - J_{m+1}(\gamma\rho)$$

and that therefore

$$\frac{\partial}{\partial x}J_0(\gamma\rho) = -\gamma J_1(\gamma\rho)\cos\omega,\quad \frac{\partial}{\partial y}J_0(\gamma\rho) = -\gamma J_1(\gamma\rho)\operatorname{sen}\omega$$

$$\frac{\partial}{\partial x}\left[\cos m\omega\, J_m(\gamma\rho)\right] = \frac{\gamma}{2}\left[\cos(m-1)\omega\, J_{m-1}(\gamma\rho) - \cos(m+1)\omega\, J_{m+1}(\gamma\rho)\right];$$

$$\frac{\partial}{\partial y}\left[\cos m\omega\, J_m(\gamma\rho)\right] = -\frac{\gamma}{2}\left[\operatorname{sen}(m-1)\omega\, J_{m-1}(\gamma\rho) + \operatorname{sen}(m+1)\omega\, J_{m+1}(\gamma\rho)\right]$$

$$\frac{\partial}{\partial x}\left[\operatorname{sen} m\omega\, J_m(\gamma\rho)\right] = \frac{\gamma}{2}\left[\operatorname{sen}(m-1)\omega\, J_{m-1} - \operatorname{sen}(m+1)\omega\, J_{m+1}\right]$$

$$\frac{\partial}{\partial y}\left[\operatorname{sen} m\omega\, J_m(\gamma\rho)\right] = \frac{\gamma}{2}\left[\cos(m-1)\omega\, J_{m-1} + \cos(m+1)\omega\, J_{m+1}\right].$$

in whose summations m always varies from zero to $+\infty$. We assume however, that the β's with zero subscript and the α's and β's with subscript -1 are zero. /28

We then find

$$
\left.
\begin{aligned}
a_0 &= \frac{\gamma}{2}\left(\alpha_1 + \beta_1^{(1)} - 2\,\alpha_0^{(2)}\right),\\
a_1 &= \frac{\gamma}{2}\left(\alpha_2 - 2\,\alpha_0 + \beta_2^{(1)} + \beta_0^{(1)} - 2\,\alpha_1^{(2)}\right),\\
b_1 &= \frac{\gamma}{2}\left(\beta_2 - \beta_0 - \alpha_2^{(1)} - 2\,\alpha_0^{(1)} - 2\,\beta_1^{(2)}\right),\\
a_m &= \frac{\gamma}{2}\left(\alpha_{m+1} - \alpha_{m-1} + \beta_{m+1}^{(1)} + \beta_{m-1}^{(1)} - 2\,\alpha_m^{(2)}\right),\\
b_m &= \frac{\gamma}{2}\left(\beta_{m+1} - \beta_{m-1} - \alpha_{m+1}^{(1)} - \alpha_{m-1}^{(1)} - 2\,\beta_m^{(2)}\right);
\end{aligned}
\right\} \qquad (21)
$$

while $\bar{a}_m$ and $\bar{b}_m$ are given by the same formulas, provided that strokes are added to all the α's and β's and the signs are changed on the last terms.

When U, V, W are given expanded into series in form (20) in S, and the functions a_m, b_m; $\bar{a}_m$, $\bar{b}_m$ are understood to be determined by means of the functions α and β which enter into U, V, W by expressions (19) and (21), the solution of our problem may also be represented by the following formulas:

$$
\left.
\begin{aligned}
u = U &+ \frac{\lambda+\mu}{8\pi\mu}\int_0^\infty \gamma\, d\gamma\, \frac{2he^{2h\gamma}(e^{-\gamma z} - e^{\gamma z}) + z(e^{2h\gamma}-1)(e^{\gamma(2h-z)} + e^{\gamma z})}{(e^{2h\gamma}-1)^2}\Big\{-a_0 \cos\omega\, J_1(\gamma\rho) +\\
&+ \sum_m \left[(a_{m+1} - a_{m-1})\cos m\omega + (b_{m+1} - b_{m-1})\,\mathrm{sen}\, m\omega\right] J_m(\gamma\rho)\Big\} +\\
&+ \frac{\lambda+\mu}{8\pi\mu}\int_0^\infty \gamma\, e^{h\gamma}\, d\gamma\, \frac{2h(e^{\gamma z} - e^{\gamma(2h-z)}) + (h-z)(e^{2h\gamma}-1)(e^{\gamma z} + e^{-\gamma z})}{(e^{2h\gamma}-1)^2}\Big\{-\bar{a}_0 \cos\omega\, J_1(\gamma\rho) +\\
&+ \sum_m \left[(\bar{a}_{m+1} - \bar{a}_{m-1})\cos m\omega + (\bar{b}_{m+1} - \bar{b}_{m-1})\,\mathrm{sen}\, m\omega\right] J_m(\gamma\rho)\Big\},\\
v = V &+ \frac{\lambda+\mu}{8\pi\mu}\int_0^\infty \gamma\, d\gamma\, \frac{2he^{2h\gamma}(e^{-\gamma z} - e^{\gamma z}) + z(e^{2h\gamma}-1)(e^{\gamma(2h-z)} + e^{\gamma z})}{(e^{2h\gamma}-1)^2}\Big\{-a_0\,\mathrm{sen}\,\omega\, J_1(\gamma\rho) +
\end{aligned}
\right\} \qquad (22)
$$

/29

$$+\sum_m [(b_{m+1}+b_{m-1})\cos m\omega-(a_{m+1}+a_{m-1})\operatorname{sen} m\omega]\,J_m(\gamma\rho)\Big\}+$$

$$+\frac{\lambda+\mu}{8\pi\mu}\int_0^\infty \gamma e^{h\gamma}d\gamma\,\frac{2h(e^{\gamma z}-e^{\gamma(2h-z)})+(h-z)(e^{2h\gamma}-1)(e^{\gamma z}+e^{-\gamma z})}{(e^{2h\gamma}-1)^2}\Big\{-\bar{a}_0\operatorname{sen}\omega\,J_1(\gamma\rho)+$$

$$+\sum_m [(\bar{b}_{m+1}+\bar{b}_{m-1})\cos m\omega-(\bar{a}_{m+1}+\bar{a}_{m-1})\operatorname{sen} m\omega]\,J_m(\gamma\rho)\Big\},$$

$$w=W-\frac{\lambda+\mu}{2\mu}z\theta+\frac{\lambda+\mu}{4\pi\mu}h\sum_m\int_0^\infty\frac{\gamma e^{h\gamma}(e^{\gamma z}-e^{-\gamma z})}{e^{2h\gamma}-1}(\bar{a}_m\cos m\omega+$$

$$+\bar{b}_m\operatorname{sen} m\omega)\,J_m(\gamma\rho)\,d\gamma, \tag{22}$$

$$\theta=\frac{1}{2\pi}\sum_m\int_0^\infty\frac{\gamma(e^{\gamma(2h-z)}-e^{\gamma z})}{e^{2h\gamma}-1}(a_m\cos m\omega+b_m\operatorname{sen} m\omega)\,J_m(\gamma\rho)\,d\gamma+$$

$$+\frac{1}{2\pi}\sum_m\int_0^\infty\gamma\frac{e^{h\gamma}(e^{\gamma z}-e^{-\gamma z})}{e^{2h\gamma}-1}(\bar{a}_m\cos m\omega+\bar{b}_m\operatorname{sen} m\omega)\,J_m(\gamma\rho)\,d\gamma,$$

in which formulas it must be assumed that $b_0 = \bar{b}_0 = a_{-1} = \bar{a}_{-1} = b_{-1} = \bar{b}_{-1} = 0$.

7. *Case in which L, M, N are given on the two limiting planes.* This new problem could be easily solved directly by starting from expressions (8) (I) and making use of the formulas which gives the function which is harmonic and regular in S by means of the values which its normal derivative acquires on σ, which has been constructed for No. 3. It is perhaps more convenient to make the problem depend on the preceding problem by observing that the indefinite equations derived with respect to z give us

$$\Delta^2\frac{\partial u}{\partial z}+\frac{\lambda+\mu}{\mu}\frac{\partial}{\partial x}\frac{\partial\theta}{\partial z}=0,\ldots \tag{23}$$

Writing the formulas similar to expression (13) for these equations and noting that the functions which are harmonic and regular in S -- which on σ assume the values $\frac{\partial u}{\partial\zeta}, \frac{\partial v}{\partial\zeta}, \frac{\partial w}{\partial\zeta}$ for the surface conditions, i.e.,

on σ_1 :

$$L=-2\mu\left(\frac{\partial u}{\partial\zeta}-\varpi_2\right),\quad M=-2\mu\left(\frac{\partial v}{\partial\zeta}+\varpi_1\right),$$
$$N=-\lambda\theta-2\mu\frac{\partial w}{\partial\zeta},$$

and on σ_2:

$$L=2\mu\left(\frac{\partial u}{\partial\zeta}-\varpi_2\right),\quad M=2\mu\left(\frac{\partial v}{\partial\zeta}+\varpi_1\right),\quad N=\lambda\theta+2\mu\frac{\partial w}{\partial\zeta},$$

are identical, respectively, to /30

$$\frac{\mathfrak{L}}{2\mu}+\varpi_2,\quad\frac{\mathfrak{M}}{2\mu}-\varpi_1,\quad\frac{\mathfrak{N}}{2\mu}-\frac{\lambda}{2\mu}\theta,$$

where $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ indicate the functions which are harmonic and regular in S which assume on σ_1 the values of L, M, N given on this plane with changed sign and on σ_2 the values of L, M, N given on σ_2 with their own sign, we obtain

$$
\left.
\begin{aligned}
\frac{\partial u}{\partial z} &= \frac{\mathfrak{L}}{2\mu} + \varpi_2 + \frac{\lambda+\mu}{4\pi\mu}\Big\{ z\frac{\partial}{\partial x}\int_{\sigma_1}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r} + \sum_1^\infty {}_n(2nh+z)\frac{\partial}{\partial x}\int_{\sigma_1}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r_{2n}} - \\
&- \sum_1^\infty {}_n(2nh-z)\frac{\partial}{\partial x}\int_{\sigma_1}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r'_{2n-1}} + (h-z)\frac{\partial}{\partial x}\int_{\sigma_2}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r} + \\
&+ \sum_1^\infty {}_n[(2n+1)h-z]\frac{\partial}{\partial x}\int_{\sigma_2}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r'_{2n}} - \\
&- \sum_1^\infty {}_n[(2n-1)h+z]\frac{\partial}{\partial x}\int_{\sigma_2}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r_{2n-1}}\Big\},\\
\frac{\partial v}{\partial z} &= \frac{\mathfrak{M}}{2\mu} - \varpi_1 + \frac{\lambda+\mu}{4\pi\mu}\Big\{ z\frac{\partial}{\partial y}\int_{\sigma_1}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r} + \sum_1^\infty {}_n(2nh+z)\frac{\partial}{\partial y}\int_{\sigma_1}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r_{2n}} - \\
&- \sum_1^\infty {}_n(2nh-z)\frac{\partial}{\partial y}\int_{\sigma_1}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r'_{2n-1}} + (h-z)\frac{\partial}{\partial y}\int_{\sigma_2}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r} + \\
&+ \sum_1^\infty {}_n[(2n+1)h-z]\frac{\partial}{\partial y}\int_{\sigma_2}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r'_{2n}} - \\
&- \sum_1^\infty {}_n[(2n-1)h+z]\frac{\partial}{\partial y}\int_{\sigma_2}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r_{2n-1}}\Big\},\\
\frac{\partial w}{\partial z} &= \frac{\mathfrak{N}}{2\mu} - \frac{\lambda}{2\mu}\theta - \frac{\lambda+\mu}{2\mu}z\frac{\partial\theta}{\partial z} + \frac{\lambda+\mu}{4\pi\mu}h\Big\{\frac{\partial}{\partial z}\int_{\sigma_2}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r} + \\
&+ \sum_1^\infty {}_n\frac{\partial}{\partial z}\int_{\sigma_2}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r'_{2n}} + \sum_1^\infty {}_n\frac{\partial}{\partial z}\int_{\sigma_2}\frac{\partial\theta}{\partial\zeta}\frac{d\sigma}{r_{2n-1}}\Big\}.
\end{aligned}
\right\} \qquad (24)
$$

Calling ψ_0, ψ'_0, ψ_n, ψ_{-n}, ψ'_n, ψ'_{-n} the expressions similar to ϕ of the preceding problem when in the respective integrals, instead of θ, we set $\frac{\partial\theta}{\partial\zeta}$, the formulas (24) may be written /31

$$
\left.
\begin{aligned}
\frac{\partial u}{\partial z} &= \frac{\mathfrak{L}}{2\mu} + \varpi_2 + \frac{\lambda+\mu}{4\pi\mu}\sum_{-\infty}^{+\infty}{}_n\left[(2nh+z)\frac{\partial\psi_n}{\partial x} + (2nh+h-z)\frac{\partial\psi'_n}{\partial x}\right],\\
\frac{\partial v}{\partial z} &= \frac{\mathfrak{M}}{2\mu} - \varpi_1 + \frac{\lambda+\mu}{4\pi\mu}\sum_{-\infty}^{+\infty}{}_n\left[(2nh+z)\frac{\partial\psi_n}{\partial y} + (2nh+h-z)\frac{\partial\psi'_n}{\partial y}\right],\\
\frac{\partial w}{\partial z} &= \frac{\mathfrak{N}}{2\mu} - \frac{\lambda}{2\mu}\theta - \frac{\lambda+\mu}{2\mu}z\frac{\partial\theta}{\partial z} + \frac{\lambda+\mu}{4\pi\mu}h\sum_{-\infty}^{+\infty}{}_n\frac{\partial\psi'_n}{\partial z}.
\end{aligned}
\right\} \qquad (24)
$$

The problem is reduced to determining the unknown functions $\bar{\omega}_1$, $\bar{\omega}_2$ and the values of ψ, so that the following holds identically

$$
\frac{\partial\varpi_1}{\partial z} = \frac{1}{2\mu}\left(\frac{\partial\mathfrak{N}}{\partial y} - \frac{\partial\mathfrak{M}}{\partial z}\right) - \frac{\lambda}{2\mu}\frac{\partial\theta}{\partial y} - \Big\} \qquad (25)
$$

$$
\left.
\begin{aligned}
&-\frac{\lambda+\mu}{4\pi\mu}\sum_{-\infty}^{+\infty}{}_n\left[\frac{\partial}{\partial y}\left(2nh\frac{\partial\psi_n}{\partial z}+\psi_n\right)+\frac{\partial}{\partial y}\left(2nh\frac{\partial\psi'_n}{\partial z}-\psi'_n\right)\right],\\
&\qquad\frac{\partial\varpi_2}{\partial z}=\frac{1}{2\mu}\left(\frac{\partial\mathfrak{L}}{\partial z}-\frac{\partial\mathfrak{N}}{\partial x}\right)+\frac{\lambda}{2\mu}\frac{\partial\theta}{\partial x}+\\
&+\frac{\lambda+\mu}{4\pi\mu}\sum_{-\infty}^{+\infty}{}_n\left[\frac{\partial}{\partial x}\left(2nh\frac{\partial\psi_n}{\partial z}+\psi_n\right)+\frac{\partial}{\partial x}\left(2nh\frac{\partial\psi'_n}{\partial z}-\psi'_n\right)\right],\\
&\frac{\partial\theta}{\partial z}=\frac{1}{\lambda+\mu}\left(\frac{\partial\mathfrak{L}}{\partial x}+\frac{\partial\mathfrak{M}}{\partial y}+\frac{\partial\mathfrak{N}}{\partial z}\right)-\frac{h}{\pi}\sum_{-\infty}^{+\infty}{}_n\, n\frac{\partial^2}{\partial z^2}(\psi_n+\psi'_n)
\end{aligned}
\right\}\qquad(25)
$$

and so that the values of ψ satisfy the same conditions 2, 3, 4, and 5 which are satisfied by ϕ of the preceding problem.

Of equations (25), only the third is a condition for the values of ψ. When these functions are determined, the first two of equations (25) are determined by quadratures; $\bar{\omega}_1$ and $\bar{\omega}_2$ and u, v, w are also determined by expression (24) or (24') by means of quadratures.

Noting that

$$\frac{\partial\theta}{\partial z}=\frac{1}{2\pi}\sum_{-\infty}^{+\infty}{}_n\frac{\partial}{\partial z}(\psi'_n-\psi_n),\qquad(26)$$

/32

we may write the last equation in expression (25) as

$$\sum_{-\infty}^{+\infty}{}_n\frac{\partial}{\partial z}(\psi'_n-\psi_n)+2h\sum_{-\infty}^{+\infty}{}_n\, n\frac{\partial^2}{\partial z^2}(\psi_n+\psi'_n)=\frac{2\pi}{\lambda+\mu}\left(\frac{\partial\mathfrak{L}}{\partial x}+\frac{\partial\mathfrak{M}}{\partial y}+\frac{\partial\mathfrak{N}}{\partial z}\right)\qquad(25')$$

If we now set

$$
\left.
\begin{aligned}
\mathfrak{L}&=\sum{}_m\int_0^\infty e^{-\gamma z}(\alpha_m\cos m\omega+\beta_m\operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma+\\
&+\sum{}_m\int_0^\infty e^{-\gamma(h-z)}(\bar{\alpha}_m\cos m\omega+\bar{\beta}_m\operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma,\\
\mathfrak{M}&=\sum{}_m\int_0^\infty e^{-\gamma z}(\alpha^{(1)}_m\cos m\omega+\beta^{(1)}_m\operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma+\\
&+\sum{}_m\int_0^\infty e^{-\gamma(h-z)}(\bar{\alpha}^{(1)}_m\cos m\omega+\bar{\beta}^{(1)}_m\operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma,\\
\mathfrak{N}&=\sum{}_m\int_0^\infty e^{-\gamma z}(\alpha^{(2)}_m\cos m\omega+\beta^{(2)}_m\operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma+\\
&+\sum{}_m\int_0^\infty e^{-\gamma(h-z)}(\bar{\alpha}^{(2)}_m\cos m\omega+\bar{\beta}^{(2)}_m\operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma,
\end{aligned}
\right\}\qquad(27)
$$

it will be found that

$$
\left.
\begin{aligned}
\frac{\partial\mathfrak{L}}{\partial x}+\frac{\partial\mathfrak{M}}{\partial y}+\frac{\partial\mathfrak{N}}{\partial z}&=\sum{}_m\int_0^\infty e^{-\gamma z}(\mathfrak{a}_m\cos m\omega+\mathfrak{b}_m\operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma+\\
&+\sum{}_m\int_0^\infty e^{-\gamma(h-z)}(\bar{\mathfrak{a}}_m\cos m\omega+\bar{\mathfrak{b}}_m\operatorname{sen} m\omega)J_m(\gamma\rho)\,d\gamma
\end{aligned}
\right\}\qquad(28)
$$

the functions a_m, b_m; $\bar{a}_m$, $\bar{b}_m$ of γ being expressed by means of functions α and β from expression (21). If we now hypothesize:

$$\left.\begin{aligned} \psi_0 &= \sum_m \int_0^\infty e^{-\gamma z} (a_m \cos m\omega + b_m \operatorname{sen} m\omega) J_m(\gamma\rho)\, d\gamma, \\ \psi'_0 &= \sum_m \int_0^\infty e^{-\gamma(h-z)} (\bar{a}_m \cos m\omega + \bar{b}_m \operatorname{sen} m\omega) J_m(\gamma\rho)\, d\gamma \ (*), \end{aligned}\right\} \quad (29)$$

the other values of ψ will be determined, like the corresponding values of ϕ, by expression (16'). Then substituting expression (29) for ψ_0, ψ'_0 in expression (25'), similar expressions for the other values of ψ, and expression (28) for $\frac{\partial \mathfrak{L}}{\partial x} + \frac{\partial \mathfrak{M}}{\partial y} + \frac{\partial \mathfrak{N}}{\partial z}$, and setting equal the coefficients of /33

$$\begin{gathered} e^{-\gamma z} \cos m\omega\, J_m(\gamma\rho), \quad e^{-\gamma z} \operatorname{sen} m\omega\, J_m(\gamma\rho), \\ e^{-\gamma(h-z)} \cos m\omega\, J_m(\gamma\rho), \quad e^{-\gamma(h-z)} \operatorname{sen} m\omega\, J_m(\gamma\rho), \end{gathered}$$

we find the following equations:

$$\left.\begin{aligned} \frac{2\pi}{\lambda+\mu} a_m &= \gamma\left[a_m\left(1 + \sum_1^\infty{}_n e^{-2nh\gamma}\right) - \bar{a}_m \sum_1^\infty{}_n e^{-(2n-1)h\gamma}\right] + \\ &\quad + 2h\gamma^2\left[a_m \sum_1^\infty{}_n n e^{-2nh\gamma} - \bar{a}_m \sum_1^\infty{}_n n e^{-(2n-1)h\gamma}\right], \\ \frac{2\pi}{\lambda+\mu} b_m &= \gamma\left[b_m\left(1 + \sum_1^\infty{}_n e^{-2nh\gamma}\right) - \bar{b}_m \sum_1^\infty{}_n e^{-(2n-1)h\gamma}\right] + \\ &\quad + 2h\gamma^2\left[b_m \sum_1^\infty{}_n n e^{-2nh\gamma} - \bar{b}_m \sum_1^\infty{}_n n e^{-(2n-1)h\gamma}\right], \\ \frac{2\pi}{\lambda+\mu} \bar{a}_m &= \gamma\left[\bar{a}_m\left(1 + \sum_1^\infty{}_n e^{-2nh\gamma}\right) - a_m \sum_1^\infty{}_n e^{-(2n-1)h\gamma}\right] + \\ &\quad + 2h\gamma^2\left[\bar{a}_m \sum_1^\infty{}_n n e^{-2nh\gamma} - a_m \sum_1^\infty{}_n n e^{-(2n-1)h\gamma}\right], \\ \frac{2\pi}{\lambda+\mu} \bar{b}_m &= \gamma\left[\bar{b}_m\left(1 + \sum_1^\infty{}_n e^{-2nh\gamma}\right) - b_m \sum_1^\infty{}_n e^{-(2n-1)h\gamma}\right] + \\ &\quad + 2h\gamma^2\left[\bar{b}_m \sum_1^\infty{}_n n e^{-2nh\gamma} - b_m \sum_1^\infty{}_n n e^{-(2n-1)h\gamma}\right], \end{aligned}\right\} \quad (30)$$

from which we have

---

* We have indicated the indeterminate functions of γ which enter into $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$, and into ψ_0, ψ'_0, with the same name which we called the similar functions which enter into U, V. W and into ϕ_0, ϕ'_0 in the preceding problem, both for the sake of simplicity and because there is no room for any misunderstanding.

$$\left.\begin{aligned}
\Delta a_m &= (e^{2h\gamma} + 2h\gamma - 1)\, e^{h\gamma}\, \mathfrak{a}_m + [e^{2h\gamma}(2h\gamma + 1) - 1]\, \bar{\mathfrak{a}}_m, \\
\Delta \bar{a}_m &= [e^{2h\gamma}(2h\gamma + 1) - 1]\, \mathfrak{a}_m + (e^{2h\gamma} + 2h\gamma - 1)\, e^{h\gamma}\, \bar{\mathfrak{a}}_m, \\
\Delta b_m &= (e^{2h\gamma} + 2h\gamma - 1)\, e^{h\gamma}\, \mathfrak{b}_m + [e^{2h\gamma}(2h\gamma + 1) - 1]\, \bar{\mathfrak{b}}_m, \\
\Delta \bar{b}_m &= [e^{2h\gamma}(2h\gamma + 1) - 1]\, \mathfrak{b}_m + (e^{2h\gamma} + 2h\gamma - 1)\, e^{h\gamma}\, \bar{\mathfrak{b}}_m, \\
\Delta &= \frac{(\lambda+\mu)\gamma\, e^{h\gamma}}{2\pi(e^{2h\gamma} - 1)}\left[(e^{2h\gamma} - 1)^2 - 4h^2\gamma^2 e^{2h\gamma}\right].
\end{aligned}\right\} \quad (30')$$

The problem is solved, except for performing the quadratures, and verification of the solution may be readily provided by bearing in mind what has been said for the preceding problem. /34

8. To complete the analytical description of the solution, let us set

$$\left.\begin{aligned}
\bar{\mathfrak{L}} &= \sum_m \int_0^\infty \frac{e^{-\gamma z}}{\gamma}(\alpha_m \cos m\omega + \beta_m \operatorname{sen} m\omega)\, J_m(\gamma\rho)\, d\gamma + \\
&\quad + \sum_m \int_0^\infty \frac{e^{-\gamma(h-z)}}{\gamma}(\bar{\alpha}_m \cos m\omega + \bar{\beta}_m \operatorname{sen} m\omega)\, J_m(\gamma\rho)\, d\gamma, \\
\bar{\mathfrak{M}} &= -\sum_m \int_0^\infty \frac{e^{-\gamma z}}{\gamma}(\alpha_m^{(1)} \cos m\omega + \beta_m^{(1)} \operatorname{sen} m\omega)\, J_m(\gamma\rho)\, d\gamma + \\
&\quad + \sum_m \int_0^\infty \frac{e^{-\gamma(h-z)}}{\gamma}(\bar{\alpha}_m^{(1)} \cos m\omega + \bar{\beta}_m^{(1)} \operatorname{sen} m\omega)\, J_m(\gamma\rho)\, d\gamma, \\
\bar{\mathfrak{N}} &= -\sum_m \int_0^\infty \frac{e^{-\gamma z}}{\gamma}(\alpha_m^{(2)} \cos m\omega + \beta_m^{(2)} \operatorname{sen} m\omega)\, J_m(\gamma\rho)\, d\gamma + \\
&\quad + \sum_m \int_0^\infty \frac{e^{-\gamma(h-z)}}{\gamma}(\bar{\alpha}_m^{(2)} \cos m\omega + \bar{\beta}_m^{(2)} \operatorname{sen} m\omega)\, J_m(\gamma\rho)\, d\gamma,
\end{aligned}\right\} \quad (31)$$

in which formulas it is understood that the term corresponding to m = 0 is suitably modified so that the integral will be finite.*

---

* To construct the expression, for example, for $\bar{\mathfrak{L}}$, without misunderstanding, it may be remarked that $\bar{\mathfrak{L}}$ represents the harmonic function in S whose normal derivative assumes on σ the values L, and therefore use may be made of formula (10) in which the condition is made that

$$\int_{\sigma_1} L\left(\frac{1}{r_{2n}} - \frac{1}{2nh}\right) d\sigma = \int_{\sigma_1} L\left(\frac{1}{r_{2n}} - \int_0^\infty e^{-2nh\gamma}\, d\gamma\right) d\sigma \ldots;$$

It then suffices to develop $\frac{1}{r_{2n}}$ in Bessel functions, as was done by Heine (loc. cit.) and to sum with respect to n.

Similarly, let us indicate by a letter or a stroke, or with two strokes on top, every function which is harmonic and regular in S whose derivative with respect to z, or whose second derivative with respect to z, is the function which is harmonic and regular in S represented bh the same letter without strokes. We /35
will thus have

$$
\left.
\begin{aligned}
u &= \frac{\bar{\mathfrak{L}}}{2\mu} + \bar{\varpi}_2 + \frac{\lambda+\mu}{4\pi\mu}\sum_{-\infty}^{+\infty}{}_n\Big[(2nh+z)\frac{\partial\bar{\psi}_n}{\partial x} + \\
&\qquad + (2nh+h-z)\frac{\partial\bar{\psi}'_n}{\partial x} - \frac{\partial}{\partial x}(\bar{\bar{\psi}}_n - \bar{\bar{\psi}}'_n)\Big], \\
v &= \frac{\bar{\mathfrak{M}}}{2\mu} - \bar{\varpi}_1 + \frac{\lambda+\mu}{2\pi\mu}\sum_{-\infty}^{+\infty}{}_n\Big[(2nh+z)\frac{\partial\bar{\psi}_n}{\partial y} + \\
&\qquad + (2nh+h-z)\frac{\partial\bar{\psi}'_n}{\partial y} - \frac{\partial}{\partial y}(\bar{\bar{\psi}}_n - \bar{\bar{\psi}}'_n)\Big], \\
w &= \frac{\bar{\mathfrak{N}}}{2\mu} + \frac{1}{2}\bar{\theta} - \frac{\lambda+\mu}{2\mu}z\theta + \frac{\lambda+\mu}{4\pi\mu}h\sum_{-\infty}^{+\infty}{}_n\psi'_n, \ (*)
\end{aligned}
\right\} \qquad (24'')*
$$

$$
\left.
\begin{aligned}
\varpi_1 &= \frac{1}{2\mu}\left(\frac{\partial\bar{\mathfrak{N}}}{\partial y} - \frac{\partial\bar{\mathfrak{M}}}{\partial z}\right) - \frac{\lambda}{2\mu}\frac{\partial\bar{\theta}}{\partial y} - \\
&- \frac{\lambda+\mu}{4\pi\mu}\sum_{-\infty}^{+\infty}{}_n\left[\frac{\partial}{\partial y}(2nh\psi_n + \bar{\psi}_n) + \frac{\partial}{\partial y}(2nh\psi'_n - \bar{\psi}'_n)\right], \\
\varpi_2 &= \frac{1}{2\mu}\left(\frac{\partial\bar{\mathfrak{L}}}{\partial z} - \frac{\partial\bar{\mathfrak{N}}}{\partial x}\right) + \frac{\lambda}{2\mu}\frac{\partial\bar{\theta}}{\partial x} + \\
&+ \frac{\lambda+\mu}{4\pi\mu}\sum_{-\infty}^{+\infty}{}_n\left[\frac{\partial}{\partial x}(2nh\psi_n + \bar{\psi}_n) + \frac{\partial}{\partial x}(2nh\psi'_n - \bar{\psi}'_n)\right], \\
\theta &= \frac{1}{\lambda+\mu}\left(\frac{\partial\bar{\mathfrak{L}}}{\partial x} + \frac{\partial\bar{\mathfrak{M}}}{\partial y} + \frac{\partial\bar{\mathfrak{N}}}{\partial z}\right) - \\
&- \frac{h}{\pi}\sum_{-\infty}^{+\infty}{}_n n\frac{\partial}{\partial z}(\psi_n + \psi'_n) = \frac{1}{2\pi}\sum_{-\infty}^{+\infty}{}_n(\psi'_n - \psi_n).
\end{aligned}
\right\} \qquad (25')
$$

These formulas, finally -- by substituting expressions (29) and similar ones for values of ψ and performing the summation with respect to n -- may be formulated as follows /36

$$u = \frac{\bar{\psi}}{2\mu} + \bar{\varpi}_2 - \qquad (25')$$

* If the ψ_n, ψ'_n are represented by integrals extended over σ_1 or σ_2, it must be understood that in these formulas

$$\psi_n = \int_{\sigma_1}\theta\left(\frac{1}{r_{2n}} - \frac{1}{2nh}\right)d\sigma, \quad \psi_{-n} = \int_{\sigma_2}\theta\left(\frac{1}{r'_{2n-1}} - \frac{1}{2nh}\right)d\sigma\ldots$$

$$-\frac{\lambda+\mu}{8\pi\mu}\int_0^\infty d\gamma\,\frac{2h\gamma e^{2h\gamma}(e^{\gamma z}+e^{-\gamma z})+(e^{2h\gamma}-1)[e^{\gamma(2h-z)}+e^{\gamma z}+z\gamma(e^{\gamma(2h-z)}-e^{\gamma z})]}{\gamma(e^{2h\gamma}-1)^2}\Big\{-a_0\cos\omega\,J_1(\gamma\rho)+$$

$$+\sum_m[(a_{m+1}-a_{m-1})\cos m\omega+(b_{m+1}-b_{m-1})\,\mathrm{sen}\,m\omega]\,J_m(\gamma\rho)\Big\}+$$

$$+\frac{\lambda+\mu}{8\pi\mu}\int_0^\infty e^{h\gamma}d\gamma\,\frac{2h\gamma(e^{\gamma z}+e^{\gamma(2h-z)})+(e^{2h\gamma}-1)[e^{\gamma z}+e^{-\gamma z}+(h-z)\gamma(e^{\gamma z}-e^{-\gamma z})]}{\gamma(e^{2h\gamma}-1)^2}\Big\{-\bar a_0\cos\omega\,J_1(\gamma\rho)+$$

$$+\sum_m[(\bar a_{m+1}-\bar a_{m+1})\cos m\omega+(\bar b_{m+1}-\bar b_m)\,\mathrm{sen}\,m\omega)\,J_m(\gamma\rho)\Big\}.$$

$$v=\frac{\bar{\mathfrak{M}}}{2\mu}-\bar\varpi_1-$$

$$-\frac{\lambda+\mu}{8\pi\mu}\int_0^\infty d\gamma\,\frac{2h\gamma e^{2h\gamma}(e^{\gamma z}+e^{-\gamma z})+(e^{2h\gamma}-1)[e^{\gamma(2h-z)}+e^{\gamma z}+z\gamma(e^{\gamma(2h-z)}-e^{\gamma z})]}{\gamma(e^{2h\gamma}-1)^2}\Big\{-a_0\,\mathrm{sen}\,\omega\,J_1(\gamma\rho)+\qquad(25')$$

$$+\sum_m[(b_{m+1}+b_{m-1})\cos m\omega-(a_{m+1}+a_{m-1})\,\mathrm{sen}\,m\omega]\,J_m(\gamma\rho)\Big\}-$$

$$+\frac{\lambda+\mu}{8\pi\mu}\int_0^\infty e^{h\gamma}d\gamma\,\frac{2h\gamma(e^{\gamma z}+e^{\gamma(2h-z)})+(e^{2h\gamma}-1)[e^{\gamma z}+e^{-\gamma z}+(h-z)\gamma(e^{\gamma z}-e^{-\gamma z})]}{\gamma(e^{2h\gamma}-1)^2}\Big\{-\bar a_0\,\mathrm{sen}\,\omega\,J_1(\gamma\rho)+$$

$$+\sum_m[(\bar b_{m+1}+\bar b_{m-1})\cos m\omega-(\bar a_{m+1}+\bar a_{m-})\,\mathrm{sen}\,m\omega]\,J_m(\gamma\rho)\Big\},$$

$$w=\frac{\bar{\mathfrak{N}}}{2\mu}+\frac{1}{2}\bar\theta-\frac{\lambda+\mu}{2\mu}z\,\zeta+$$

$$+\frac{\lambda+\mu}{4\pi\mu}h\sum_m\int_0^\infty e^{h\gamma}\frac{e^{\gamma z}+e^{-\gamma z}}{e^{2h\gamma}-1}(\bar a_m\cos m\omega+\bar b_m\,\mathrm{sen}\,m\omega)\,J_m(\gamma\rho)\,d\gamma.$$

$$\varpi_1=\frac{1}{2\mu}\left(\frac{\partial\bar{\mathfrak{N}}}{\partial y}-\frac{\partial\bar{\mathfrak{M}}}{\partial z}\right)-\frac{\lambda}{2\mu}\frac{\partial\bar\theta}{\partial y}-$$

$$-\frac{\lambda+\mu}{8\pi\mu}\int_0^\infty d\gamma\,\frac{2h\gamma e^{2h\gamma}(e^{-\gamma z}-e^{\gamma z})+(e^{2h\gamma}-1)(e^{\gamma z}-e^{\gamma(2h-z)})}{\gamma(e^{2h\gamma}-1)^2}\Big\{-a_0\,\mathrm{sen}\,\omega\,J_1(\gamma\rho)+\qquad(25'')$$

$$+\sum_m[(b_{m+1}+b_{m-1})\cos m\omega-(a_{m+1}+a_{m-1})\,\mathrm{sen}\,m\omega]\,J_m(\gamma\rho)\Big\}-$$

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$$-\frac{\lambda+\mu}{8\pi\mu}\int_0^\infty e^{h\gamma}d\gamma\,\frac{2h\gamma e^{h\gamma}(e^{-\gamma(h-z)}-e^{\gamma(h-z)})+(e^{2h\gamma}-1)(e^{-\gamma z}-e^{\gamma z})}{\gamma(e^{2h\gamma}-1)^2}\Big\{-\bar a_0\,\mathrm{sen}\,\omega\,J_1(\gamma\rho)+$$

$$+\sum_m[(\bar b_{m+1}+\bar b_{m-1})\cos m\omega-(\bar a_{m+1}+\bar a_{m-1})\,\mathrm{sen}\,m\omega]\,J_m(\gamma\rho)\Big\},$$

$$\varpi_2=\frac{1}{2\mu}\left(\frac{\partial\bar{\mathfrak{L}}}{\partial z}-\frac{\partial\bar{\mathfrak{N}}}{\partial x}\right)+\frac{\lambda}{2\mu}\frac{\partial\bar\theta}{\partial x}+$$

$$+\frac{\lambda+\mu}{8\pi\mu}\int_0^\infty d\gamma\,\frac{2h\gamma e^{2h\gamma}(e^{-\gamma z}-e^{\gamma z})+(e^{2h\gamma}-1)(e^{\gamma z}-e^{\gamma(2h-z)})}{\gamma(e^{2h\gamma}-1)^2}\Big\{-a_0\cos\omega\,J_1(\gamma\rho)+$$

$$+\sum_m [(a_{m+1}-a_{m-1})\cos m\,\omega + (b_{m+1}-b_{m-1})\,\mathrm{sen}\, m\,\omega]\, J_m(\gamma\rho)\Big\} +$$

$$+\frac{\lambda+\mu}{8\pi\mu}\int_0^\infty e^{h\gamma}d\gamma\, \frac{2h\gamma e^{h\gamma}(e^{-\gamma(h-z)} - e^{\gamma(h-z)}) + (e^{2h\gamma}-1)(e^{-\gamma z}-e^{\gamma z})}{\gamma(e^{2h\gamma}-1)^2}\Big\{-(\bar{a}_0\, \mathrm{sen}\,\omega\, J_1(\gamma\rho)+$$

$$+\sum_m [(\bar{a}_{m+1}-\bar{a}_{m-1})\cos m\,\omega + (\bar{b}_{m+1}-\bar{b}_{m-1})\,\mathrm{sen}\, m\,\omega]\, J_m(\gamma\rho)\Big\}, \qquad (25'')$$

$$\zeta = -\frac{1}{2\pi}\sum_m \int_0^\infty \frac{e^{\gamma(2h-z)}+e^{\gamma z}}{e^{2h\gamma}-1}(a_m\cos m\,\omega + b_m\, \mathrm{sen}\, m\,\omega)\, J'_m(\gamma\rho)\,d\gamma +$$

$$+\frac{1}{2\pi}\sum_m\int_0^\infty \frac{e^{h\gamma}(e^{\gamma z}+e^{-\gamma z})}{e^{2h\gamma}-1}(\bar{a}_m \cos m\,\omega+\bar{b}_m\, \mathrm{sen}\, m\,\omega)\,J_m(\gamma\rho)\,d\gamma \ (*),$$

in which formulas it must still be assumed that $b_0 = \bar{b}_0 = a_{-1} = \bar{a}_{-1} = b_{-1} = \bar{b}_{-1} = 0$.

8. *Cases in which other conditions are given on the two limiting planes.* By the use of methods similar to the preceding ones, all the other problems in which some of the displacements and some of the stresses on σ are given may be easily solved. These problems also include the four in which both on σ_1 and on σ_2 (1) u, v, N; (2) L, v, w or u, M, w; (3) u, M, N or L, v, N; (4) L, M, w are given. To obtain the solution of these problems, it is sufficient to combine in proper fashion expressions (13) or (13') derived with respect to z, and expressions (24) or (24'), and to bear in mind that, given ϕ_0 and ϕ'_0 expanded into series, as by expression (16), the expansions into similar series of ψ_0 and ψ'_0 are /38

$$\psi_0 = \sum_m \int_0^\infty \frac{\gamma e^{-\gamma z}}{e^{2h\gamma}-1}\Big\{[-(e^{2h\gamma}+1)\,a_m + 2\,e^{h\gamma}\,\bar{a}_m]\cos m\,\omega +$$
$$+[-(e^{2h\gamma}+1)\,b_m + 2\,e^{h\gamma}\,\bar{b}_m]\,\mathrm{sen}\, m\,\omega\Big\}\,J_m(\gamma\rho)\,d\gamma,$$

$$\psi'_0 = \sum_m \int_0^\infty \frac{\gamma e^{-\gamma(h-z)}}{e^{2h\gamma}-1}\Big\{[-2\,e^{h\gamma}\,a_m + (e^{2h\gamma}+1)\,\bar{a}_m]\cos m\,\omega +$$
$$+[-2\,e^{h\gamma}\,b_m + (e^{2h\gamma}+1)\,\bar{b}_m)\,\mathrm{sen}\, m\,\omega\Big\}\,J_m(\gamma\rho)\,d\gamma, \qquad (32)$$

These values of ψ_0 and ψ'_0 are easily expressed by a series of derivatives of ϕ_0 with respect to z, and may be obtained by setting equal the coefficients of the two sides of the equation

$$\sum_{-\infty}^{+\infty}{}_n \frac{\partial^2}{\partial z^2}(\varphi'_n - \varphi_n) = \sum_{-\infty}^{+\infty}{}_n \frac{\partial}{\partial z}(\psi'_n - \psi_n).$$

(*) In this formula, we must note that a_m, b_m, $\bar{a}_m$, $\bar{b}_m$ have a factor γ.

In addition to the four preceding problems, twenty other problems may be solved with equal ease, i.e., those in which on one of the planes, for example, u, v, w or L, M, N are given on σ_1, or one of the other six combinations enumerated above, while on σ_2 other conditions are given, always taken from these combinations, but not identical to those given on σ_1. To solve these other problems, the formula must also be used which gives the function, harmonic in S, in terms of the values which this function assumes on one of the planes and by means of the values which its normal derivative assumes on the other plane, and of the corresponding Green function of which we spoke in No. 2.

Among all of these problems, special mention should be made of the three in which the normal component of the stress and the tangential components of the displacements on any one of the planes is given, or vice versa, because to solve them it is not necessary to have recourse to expansions of special functions into series. Availing ourselves, in fact, of one property of the elasticity equations taken from Professor Somigliana* we may -- by starting with the solutions of similar problems given on page 13 of Note I and relative to the case in which the elastic body is limited by a single plane -- formulate infinite solutions of the elasticity problem with successive reflections, in Professor Somigliana's sense, with respect to planes σ_1 and σ_2. With these infinite solutions, the solutions of our problems may be formulated by an alternate method. /39

II. Problems in Which the Elastic Body is Limited by Two Concentric Spheres

1. *Green's Function and Harmonic Function.* About this very well-known topic we will say as much as is necessary in order to proceed safely in our computations. Let us assume that the two spheres limiting the elastic body have as their common center the coordinate origin, and let us indicate their surfaces by σ_1 and σ_2, and their radii by R_1, R_2, respectively, and let $R_1 < R_2$. Meantime, we shall continue to indicate by σ the combination of σ_1 and σ_2, and by S the poriton of space limited by them.

If $A \equiv (x, y, z)$ is a point inside S and (ξ, η, ζ) any other point of S, let us set

$$l = \sqrt{x^2 + y^2 + z^2}, \quad \rho = \sqrt{\xi^2 + \eta^2 + \zeta^2},$$

$$l \rho \cos \omega = x \xi + y \eta + z \zeta, \quad r = \sqrt{l^2 + \rho^2 - 2 l \rho \cos \omega}$$

which is the notation already adopted. Together with point A, let us consider the infinite series of points $A_1, A_2, \ldots, A_n, \ldots$ which are obtained by taking in succession the reciprocals of A first with respect to σ_1 and then with

* "On the Principle of the Images of Lord Kelvin and the Elasticity Equations", Rediconti dell'Accademia dei Lincei, Vol. 11, 1st half-year, signature 5, fascicle 4.

respect to σ_2, which will alternately fall on the outside of σ_2 and on the inside of σ_1. Let us consider the infinite series of points $A'_1, A'_2, \ldots A'_n, \ldots$, which are obtained by taking in succession the reciprocals of A first with respect to σ_1 and then with respect to σ_2, which will alternately fall on the outside of σ_2 and on the inside of σ_1. All these points, as is known, are located on the radius vector which goes to point A. Let us then designate l_i and r_i the distances from the origin and from point (ξ, η, ζ) to A_i, and by l'_i, r'_i the distance of these same two points from A'_i. We will clearly have /40

$$\left.\begin{aligned}
&l_1 = \frac{R_1^2}{l},\ l_2 = \left(\frac{R_2}{R_1}\right)^2 l, \ldots,\ l_{2n} = \left(\frac{R_2}{R_1}\right)^{2n} l,\ l_{2n+1} = \left(\frac{R_1}{R_2}\right)^{2n} \frac{R_1^2}{l}, \ldots \\
&l'_1 = \frac{R_2^2}{l},\ l'_2 = \left(\frac{R_1}{R_2}\right)^2 l, \ldots,\ l'_{2n} = \left(\frac{R_1}{R_2}\right)^{2n} l,\ l'_{2n+1} = \left(\frac{R_2}{R_1}\right)^{2n} \frac{R_2^2}{l}, \ldots \\
&r_i = \sqrt{\rho^2 + l_i^2 - 2 l_i \rho \cos\omega},\quad r'_i = \sqrt{l'^2_i + \rho^2 - 2 l'_i \rho \cos\omega}.
\end{aligned}\right\} \tag{33}$$

By known properties of the inversion, when point (ξ, η, ζ) is on σ_1, we have

$$\begin{aligned}
&\frac{1}{r} = \frac{R_1}{l}\frac{1}{r_1}, \ldots,\quad \frac{1}{r_{2n}} = \frac{R_1}{l_{2n}}\frac{1}{r_{2n+1}}, \ldots; \\
&\frac{1}{r'_1} = \frac{R_1}{l'_1}\frac{1}{r'_2}, \ldots,\quad \frac{1}{r'_{2n-1}} = \frac{R_1}{l'_{2n-1}}\frac{1}{r'_{2n}}, \ldots;
\end{aligned}$$

while if point (ξ, η, ζ) is located on σ_2 we have

$$\begin{aligned}
&\frac{1}{r} = \frac{R_2}{l}\frac{1}{r'_1}, \ldots,\quad \frac{1}{r'_{2n}} = \frac{R_2}{l'_{2n}}\frac{1}{r'_{2n+1}}, \ldots; \\
&\frac{1}{r_1} = \frac{R_2}{l_1}\frac{1}{r_2}, \ldots,\quad \frac{1}{r_{2n-1}} = \frac{R_2}{l_{2n-1}}\frac{1}{r_{2n}}, \ldots
\end{aligned}$$

If -- when point (ξ, η, ζ) is chosen in any way -- it is A which falls on σ_1 and hence $l = R_1$, we have

$$\begin{aligned}
&l = l_1,\quad l_2 = l'_1, \ldots,\quad l_i = l'_{i-1}, \ldots; \\
&r = r_1,\quad r_2 = r'_1, \ldots,\quad r_i = r'_{i-1}, \ldots,
\end{aligned}$$

If A falls on σ_2 and hence $l = R_2$, we have

$$\begin{aligned}
&l = l'_1,\quad l'_2 = l_1, \ldots,\quad l'_i = l_{i-1}, \ldots; \\
&r = r'_1,\quad r'_2 = r_1, \ldots,\quad r'_i = r_{i-1}, \ldots
\end{aligned}$$

Series

$$\left.\begin{aligned}
g &= \frac{R_1}{l}\frac{1}{r_1} - \frac{R_2}{R_1}\left(\frac{1}{r_2} - \frac{R_1}{l_2}\frac{1}{r_3}\right) - \left(\frac{R_2}{R_1}\right)^2\left(\frac{1}{r_4} - \frac{R_1}{l_4}\frac{1}{r_5}\right) - \cdots = \\
&= \frac{R_1}{l}\frac{1}{r_1} - \sum_1^{\infty} \left(\frac{R_2}{R_1}\right)^n \left(\frac{1}{r_{2n}} - \frac{R_1}{l_{2n}}\frac{1}{r_{2n+1}}\right) = \\
&= \frac{R_1}{l}\left[\frac{1}{r_1} - \frac{R_2}{l_1}\frac{1}{r_2} + \frac{R_1}{R_2}\left(\frac{1}{r_3} - \frac{R_2}{l_3}\frac{1}{r_4}\right) + \cdots\right] =
\end{aligned}\right\} \tag{34}$$

$$=\frac{R_1}{l}\sum_1^{\infty}{}_n\left(\frac{R_1}{R_2}\right)^{n-1}\left(\frac{1}{r_{2n-1}}-\frac{R_2}{l_{2n-1}}\frac{1}{r_{2n}}\right) \tag{34}$$

is such that its terms, considered in S as functions of ξ, η, ζ, are all of the same sign and are harmonic and regular functions. Based on what has been noted, it reduces to $\frac{1}{r}$ on σ_1, while on σ_2 it becomes identically zero. The same series therefore represents a function which is harmonic and regular in S and differentiable term by term any number of times. Series /41

$$\left.\begin{aligned} g' &= \frac{R_2}{l}\frac{1}{r'_1}-\frac{R_1}{R_2}\left(\frac{1}{r'_2}-\frac{R_2}{l_2}\frac{1}{r'_3}\right)-\left(\frac{R_1}{R_2}\right)^2\left(\frac{1}{r'_4}-\frac{R_2}{l'_4}\frac{1}{r'_5}\right)-\dots= \\ &= \frac{R_2}{l}\frac{1}{r'_1}-\sum_1^{\infty}{}_n\left(\frac{R_1}{R_2}\right)^n\left(\frac{1}{r'_{2n}}-\frac{R_2}{l'_{2n}}\frac{1}{r'_{2n+1}}\right)= \\ &= \frac{R_2}{l}\sum_1^{\infty}{}_n\left(\frac{R_2}{R_1}\right)^{n-1}\left(\frac{1}{r'_{2n-1}}-\frac{R_1}{l'_{2n-1}}\frac{1}{r'_{2n}}\right). \end{aligned}\right\} \tag{35}$$

possesses similar properties; g' becomes identically zero on σ_1, while on σ_2 it becomes equal to $\frac{1}{r}$. Thus Green's function relative to point A and to space S is as follows in the most suitable form for our calculations

$$G=\frac{1}{r}-g-g'. \tag{36}$$

The normal derivative of G on σ_1 is:

$$\left.\begin{aligned} -\left(\frac{\partial G}{\partial \rho}\right)_{\rho=R_1} &= \frac{1}{R_2}\left[\frac{R_2^2-l^2}{r^3}+\sum_1^{\infty}{}_n\left(\frac{R_1}{R_2}\right)^n\frac{R_2^2-l'^2_{2n}}{r'^3_{2n}}-\right. \\ &\left.-\frac{R_1}{l}\sum_1^{\infty}{}_n\left(\frac{R_1}{R_2}\right)^{n-1}\frac{R_2^2-l'^2_{2n-1}}{r'^3_{2n-1}}\right]_{\rho=R_2}. \end{aligned}\right\} \tag{37}$$

and on σ_2:

$$\left.\begin{aligned} \left(\frac{\partial G}{\partial \rho}\right)_{\rho=R_1} &= -\frac{1}{R_1}\left[\frac{R_1^2-l^2}{r^3}+\sum_1^{\infty}{}_n\left(\frac{R_2}{R_1}\right)^n\frac{R_1^2-l_{2n}^2}{r_{2n}^3}-\right. \\ &\left.-\frac{R_2}{l}\sum_1^{\infty}{}_n\left(\frac{R_2}{R_1}\right)^{n-1}\frac{R_1^2-l'^2_{2n-1}}{r'^3_{2n-1}}\right]_{=R_1} \end{aligned}\right\} \tag{37'}$$

Function Φ which is harmonic and regular in S, which assumes the values given on σ, will therefore be represented by formula /42

$$\left.\begin{aligned} 4\pi\Phi = &-\frac{1}{R_1}\left[(R_1^2-l^2)\int_{\sigma_1}\Phi\frac{d\sigma}{r^3}+\sum_1^{\infty}{}_n\left(\frac{R_2}{R_1}\right)^n(R_1^2-l_{2n}^2)\int_{\sigma_1}\Phi\frac{d\sigma}{r_{2n}^3}-\right. \\ &\left.-\frac{R_2}{l}\sum_1^{\infty}{}_n\left(\frac{R_2}{R_1}\right)^{n-1}(R_1^2-l^2_{2n-1})\int_{\sigma_1}\Phi\frac{d\sigma}{r'^3_{2n-1}}\right]+ \\ &+\frac{1}{R_2}\left[(R_2^2-l^2)\int_{\sigma_2}\Phi\frac{d\sigma}{r^3}+\sum_1^{\infty}{}_n\left(\frac{R_1}{R_2}\right)^n(R_2^2-l'^2_{2n})\int_{\sigma_2}\Phi\frac{d\sigma}{r'^3_{2n}}-\right. \\ &\left.-\frac{R_1}{l}\sum_1^{\infty}{}_n\left(\frac{R_1}{R_2}\right)^{n-1}(R_2^2-l^2_{2n-1})\int_{\sigma_2}\Phi\frac{d\sigma}{r^3_{2n-1}}\right]. \end{aligned}\right\} \tag{38}$$

2. Noting that, if Φ is a harmonic function such as $l\frac{\partial \Phi}{\partial l}$, it immediately results that in order to formulate the function which is harmonic and regular in S whose normal derivative assumes assigned values on σ, it is sufficient to formulate the function which is harmonic and regular in S which assumes on σ_1 the values given for the normal derivative on this surface multiplied by R_1, and on σ_2 the values given on σ_2 for the normal derivative, these values being similarly multiplied by R_2. From the function thus formulated, the desired function will be derived by quadrature. So we have

$$\left.\begin{aligned}4\pi l\frac{\partial \Phi}{\partial l} = &-(R_1^2 - l^2)\int_{\sigma_1}\frac{d\Phi}{dn}\frac{d\sigma}{r^3} - \sum_1^{\infty} n\left(\frac{R_2}{R_1}\right)^n (R_1^2 - l_{2n}^2)\int_{\sigma_1}\frac{d\Phi}{dn}\frac{d\sigma}{r_{2n}^3} + \\ &+\frac{R_2}{l}\sum_1^{\infty} n\left(\frac{R_2}{R_1}\right)^{n-1}(R_1^2 - l'^2_{2n-1})\int_{\sigma_1}\frac{d\Phi}{dn}\frac{d\sigma}{r'^3_{2n-1}} - \\ &-(R_2^2 - l^2)\int_{\sigma_2}\frac{d\Phi}{dn}\frac{d\sigma}{r^3} - \sum_1^{\infty} n\left(\frac{R_1}{R_2}\right)^n (R_2^2 - l'^2_{2n})\int_{\sigma_2}\frac{d\Phi}{dn}\frac{d\sigma}{r'^3_{2n}} + \\ &+\frac{R_1}{l}\sum_1^{\infty} n\left(\frac{R_1}{R_2}\right)^{n-1}(R_2^2 - l^2_{2n-1})\int_{\sigma_2}\frac{d\Phi}{dn}\frac{d\sigma}{r^3_{2n-1}}\end{aligned}\right\}$$

and thus /43

$$\left.\begin{aligned}4\pi\Phi = \text{cost.} &-\int_{+\infty}^{l}\frac{dl}{l}\left[(R_1^2 - l^2)\int_{\sigma_1}\frac{d\Phi}{dn}\frac{d\sigma}{r^3} + \right. \\ &+\sum_1^{\infty} n\left(\frac{R_2}{R_1}\right)^n (R_1^2 - l_{2n}^2)\int_{\sigma_1}\frac{d\Phi}{dn}\frac{d\sigma}{r_{2n}^3} - \\ &\left. -\frac{R_1}{l}\sum_1^{\infty} n\left(\frac{R_1}{R_2}\right)^{n-1}(R_2^2 - l^2_{2n-1})\int_{\sigma_2}\frac{d\Phi}{dn}\frac{d\sigma}{r^3_{2n-1}}\right] - \\ -\int_0^{l}\frac{dl}{l}&\left[(R_2^2 - l^2)\int_{\sigma_2}\frac{d\Phi}{dn}\frac{d\sigma}{r^3} + \sum_1^{\infty} n\left(\frac{R_1}{R_2}\right)^n (R_2^2 - l'^2_{2n})\int_{\sigma_2}\frac{d\Phi}{dn}\frac{d\sigma}{r'^3_{2n}} - \right. \\ &\left. -\frac{R_2}{l}\sum_1^{\infty} n\left(\frac{R_2}{R_1}\right)^{n-1}(R_1^2 - l'^2_{2n-1})\int_{\sigma_1}\frac{d\Phi}{dn}\frac{d\sigma}{r'^3_{2n-1}}\right].\end{aligned}\right\} \quad (39)$$

The first of the integrals which appear in this formula is certainly finite. In order to demonstrate the fact that the second is also finite, let us note that

$$\lim_{l=0}\left[(R_2^2 - l^2)\int_{\sigma_2}\frac{d\Phi}{dn}\frac{d\sigma}{r^3} + \sum_1^{\infty} n\left(\frac{R_1}{R_2}\right)^n (R_2^2 - l_{2n}^2)\int_{\sigma_2}\frac{d\Phi}{dn}\frac{d\sigma}{r_{2n}^3}\right] =$$

$$= \frac{1}{R_2}\left[1 + \sum_1^{\infty} n\left(\frac{R_1}{R_2}\right)^n\right]\int_{\sigma_2}\frac{d\Phi}{dn}d\sigma,$$

$$-\lim_{l=0}\frac{R_2}{l}\sum_1^{\infty}{}_n\left(\frac{R_2}{R_1}\right)^{n-1}(R_1^2-l'^2_{2n-1})\int_{\sigma_1}\frac{d\Phi}{dn}\frac{d\sigma}{r'^3_{2n-1}}=$$

$$=\frac{1}{R_2}\left[\sum_1^{\infty}{}_n\left(\frac{R_1}{R_2}\right)^{n-1}\right]\int_{\sigma_1}\frac{d\Phi}{dn}\,d\sigma=\frac{1}{R_2}\left[1+\sum_1^{\infty}{}_n\left(\frac{R_1}{R_2}\right)^{n}\right]\int_{\sigma_1}\frac{d\Phi}{dn}\,d\sigma$$

and that for the existence of the desired function it must be assumed that

$$\int_{\sigma_1}\frac{d\Phi}{dn}\,d\sigma+\int_{\sigma_2}\frac{d\Phi}{dn}\,d\sigma=0.$$

3. For greater clarity in future computations, let us add the following observations.

Meanwhile /44

$$(R_1^2-l^2_{2n})\int_{\sigma_1}\Phi\frac{d\sigma}{r^3_{2n}}=$$

$$=\left(\frac{R_1}{R_2}\right)^{2n}\left[\left(\frac{R_1}{R_2}\right)^{4n}R_1^2-l^2\right]\int_{\sigma_1}\Phi\frac{d\sigma}{\sqrt{\left(\frac{R_1}{R_2}\right)^{4n}R_1^2+l^2-2\left(\frac{R_1}{R_2}\right)^{2n}R_1\,l\cos\omega}^{\,3}}=$$

$$=\left(\frac{R_1}{R_2}\right)^{2n}\left[\left(\frac{R_1}{R_2}\right)^{4n}R_1^2-l^2\right]\int_{\sigma_1}\Phi\frac{d\sigma}{\mathbf{r}^3_{2n}},$$

$$\frac{R_2}{l}(R_1^2-l'^2_{2n-1})\int_{\sigma_1}\Phi\frac{d\sigma}{r'^3_{2n-1}}=$$

$$=\frac{R_2}{R_1}\left[l^2-\left(\frac{R_2}{R_1}\right)^{4n}R_1^2\right]\int_{\sigma_1}\Phi\frac{d\sigma}{\sqrt{\left(\frac{R_2}{R_1}\right)^{4n}R_1^2+l^2-2\left(\frac{R_2}{R_1}\right)^{2n}R_1\,l\cos\omega}^{\,3}}=$$

$$=\frac{R_2}{R_1}\left[l^2-\left(\frac{R_2}{R_1}\right)^{4n}R_1^2\right]\int_{\sigma_1}\Phi\frac{d\sigma}{\mathbf{r}'^3_{2n-1}},$$

$$(R_2^2-l'^2_{2n})\int_{\sigma_2}\Phi\frac{d\sigma}{r'^3_{2n}}=\left(\frac{R_2}{R_1}\right)^{2n}\left[\left(\frac{R_2}{R_1}\right)^{4n}R_2^2-l^2\right]\int_{\sigma_2}\Phi\frac{d\sigma}{\mathbf{r}'^3_{2n}},$$

$$\frac{R_1}{l}(R_2^2-l^2_{2n-1})\int_{\sigma_2}\Phi\frac{d\sigma}{r^3_{2n-1}}=\frac{R_1}{R_2}\left[l^2-\left(\frac{R_1}{R_2}\right)^{4n}R_2^2\right]\int_{\sigma_2}\Phi\frac{d\sigma}{\mathbf{r}^3_{2n-1}},$$

where we have set

$$\mathbf{r}_{2n}=\sqrt{\left(\frac{R_1}{R_2}\right)^{4n}R_1^2+l^2-2\left(\frac{R_1}{R_2}\right)^{2n}R_1\,l\cos\omega},$$

$$\mathbf{r}'_{2n-1}=\sqrt{\left(\frac{R_2}{R_1}\right)^{4n}R_1^2+l^2-2\left(\frac{R_2}{R_1}\right)^{2n}R_1\,l\cos\omega},$$

$$\mathrm{r}'_{2n} = \sqrt{\left(\frac{R_2}{R_1}\right)^{4n} R_2^2 + l^2 - 2\left(\frac{R_2}{R_1}\right)^{2n} R_2\, l \cos\omega},$$

$$\mathrm{r}_{2n-1} = \sqrt{\left(\frac{R_1}{R_2}\right)^{4n} R_2^2 + l^2 - 2\left(\frac{R_1}{R_2}\right)^{2n} R_2\, l \cos\omega}.$$

With these positions, expression (38) may be written /45

$$\begin{aligned}\Phi = \frac{1}{4\pi R_1} \Big\{ & (l^2 - R_1^2) \int_{\sigma_1} \Phi \frac{d\sigma}{r^3} + \sum_1^{\infty}{}_n \left(\frac{R_1}{R_2}\right)^n \left[l^2 - \left(\frac{R_1}{R_2}\right)^{4n} R_1^2 \right] \int_{\sigma_1} \Phi \frac{d\sigma}{\mathrm{r}_{2n}^3} - \\ & - \sum_1^{\infty}{}_n \left(\frac{R_2}{R_1}\right)^n \left[\left(\frac{R_2}{R_1}\right)^{4n} R_1^2 - l^2 \right] \int_{\sigma_1} \Phi \frac{d\sigma}{\mathrm{r}'^3_{2n-1}} \Big\} + \\ + \frac{1}{4\pi R_2} \Big\{ & (R_2^2 - l^2) \int_{\sigma_2} \Phi \frac{d\sigma}{r^3} + \sum_1^{\infty}{}_n \left(\frac{R_2}{R_1}\right)^n \left[\left(\frac{R_2}{R_1}\right)^{4n} R_2^2 - l^2 \right] \int_{\sigma_2} \Phi \frac{d\sigma}{\mathrm{r}'^3_{2n}} - \\ & - \sum_1^{\infty}{}_n \left(\frac{R_1}{R_2}\right)^n \left[l^2 - \left(\frac{R_1}{R_2}\right)^{4n} R_2^2 \right] \int_{\sigma_2} \Phi \frac{d\sigma}{\mathrm{r}^3_{2n-1}} \Big\}. \end{aligned} \qquad (38')$$

Secondly, let us remember that every function which is harmoinc and regular inside a sphere having its center at the origin may be represented by a series which is absolutley convergent and, of equal degree inside the sphere. This series has the form

$$\sum_0^{\infty}{}_m X_m(x, y, z)$$

where $X_m(x, y, z)$ is a harmonic function homogeneous in x, y, z of whole and positive degree *m*. Similarly, every function which is harmonic and regular outside a sphere having its center in the origin may be represented by a series of form

$$\sum_0^{\infty}{}_m X_{-(m+1)}(x, y, z)$$

where $X_{-(m+1)}(x, y, z)$ is a function harmonic, homogeneous in x, y, z of the whole negative -(m + 1) degree. Finally, every function harmonic and regular in the space between two concentric spheres having their common center in the coordinate origin may be represented by the sum of the two series

$$\sum_0^{\infty}{}_m X_m(x, y, z) + \sum_0^{\infty}{}_m X_{-(m+1)}(x, y, z) = \sum_{-\infty}^{+\infty}{}_m X_m(x, y, z)$$

where X_m and X_{-m} retain their former meaning. If, on the contrary, we set

$$\left.\begin{aligned} \frac{l^2 - R_1^2}{4\pi R_1} \int_{\sigma_1} \Phi \frac{d\sigma}{r^3} &= \sum_0^{\infty}{}_m X_{-(m+1)}(x, y, z), \\ \frac{R_2^2 - l^2}{4\pi R_2} \int_{\sigma_2} \Phi \frac{d\sigma}{r^3} &= \sum_0^{\infty}{}_m X_m(x, y, z), \end{aligned}\right\} \qquad (40)$$

function Φ given by expression (38') will also be represented by formula /46

$$\Phi = \sum_0^{\infty}{}_m \frac{R_2^{2m+1} - l^{2m+1}}{R_2^{2m+1} - R_1^{2m+1}} X_{-(m+1)} + \sum_0^{\infty}{}_m \frac{R_2^{2m+1}}{l^{2m+1}} \frac{l^{2m+1} - R_1^{2m+1}}{R_2^{2m+1} - R_1^{2m+1}} X_m. \qquad (38'')$$

Let us note, finally, that function $l\,\frac{\partial\Phi}{\partial l}$ may be constructed directly by means of the values which this function assumes on σ, or it may be derived by performing operation $l\,\frac{\partial}{\partial l}$ on expressions (38') or (38''). Comparing the series expansions of $l\,\frac{\partial\Phi}{\partial l}$ which are thus obtained, it is immediately found that

$$\left.\begin{aligned}
&\frac{l^2-R_1^2}{4\pi}\int_{\sigma_1}\frac{\partial\Phi}{\partial\rho}\,\frac{d\sigma}{r^3}=\\
&=\sum_0^{\infty}{}_m\frac{(2m+1)(R_1R_2)^{2m+1}l^{-2m+1}X_m-[(m+1)R_2^{2m+1}+mR_1^{2m+1}]X_{-m+1}}{R_2^{2m+1}-R_1^{2m+1}},\\
&\frac{R_2^2-l^2}{4\pi}\int_{\sigma_2}\frac{\partial\Phi}{\partial\rho}\,\frac{d\sigma}{r^3}=\\
&=\sum_0^{\infty}{}_m\frac{[(m+1)R_1^{2m+1}+mR_2^{2m+1}]X_m-(2m+1)l^{2m+1}X_{-m+1}}{R_2^{2m+1}-R_1^{2m+1}}
\end{aligned}\right\}\qquad(41)$$

the values of X entering into this formula always being those which appear in expression (40). With the aid of expressions (40) and (41), it is easy to find analytical expressions for the function, harmonic in S, which assumes given values on one of the spherical surfaces, while on the other the normal derivative assumes given values.

4. *Case in Which u, v, w, are Given on the Two Limiting Spheres.* Let us call U, V, W the functions harmonic and regular in S which on σ acquire the respective values u, v, w, and let us note that

$$\left.\begin{aligned}
&(l^2-R_1^2)\int_{\sigma_1}\xi\,\theta\,\frac{d\sigma}{r^3}=x\,(l^2-R_1^2)\int_{\sigma_1}\theta\,\frac{d\sigma}{r^3}+(l^2-R_1^2)\,\frac{\partial}{\partial x}\int_{\sigma_1}\theta\,\frac{d\sigma}{r},\\
&\left(\frac{R_1}{R_2}\right)^{2n}\left[l^2-\left(\frac{R_1}{R_2}\right)^{4n}R_1^2\right]\int_{\sigma_1}\xi\,\theta\,\frac{d\sigma}{\mathbf{r}_{2n}^3}=x\left[l^2-\left(\frac{R_1}{R_2}\right)^{4n}R_1^2\right]\int_{\sigma_1}\theta\,\frac{d\sigma}{\mathbf{r}_{2n}^3}+\\
&\qquad+\left[l^2-\left(\frac{R_1}{R_2}\right)^{4n}R_1^2\right]\frac{\partial}{\partial x}\int_{\sigma_1}\theta\,\frac{d\sigma}{\mathbf{r}_{2n}},\\
&\left(\frac{R_2}{R_1}\right)^{2n}\left[\left(\frac{R_2}{R_1}\right)^{4n}R_1^2-l^2\right]\int_{\sigma_1}\xi\,\theta\,\frac{d\sigma}{\mathbf{r}'^3_{2n-1}}=x\left[\left(\frac{R_2}{R_1}\right)^{4n}R_1^2-l^2\right]\int_{\sigma_1}\theta\,\frac{d\sigma}{\mathbf{r}'^3_{2n-1}}+\\
&\qquad+\left[\left(\frac{R_2}{R_1}\right)^{4n}R_1^2-l^2\right]\frac{\partial}{\partial x}\int_{\sigma_1}\theta\,\frac{d\sigma}{\mathbf{r}'_{2n-1}},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\}\qquad(42)$$

where the unwritten formulas relate to the similar integrals extended over σ_2 /47
and to those in which ξ changes successively into η and ζ. If, for purposes of brevity, we then set

$$\left.\begin{aligned}
&4\pi R_1\theta_0=(l^2-R_1^2)\int_{\sigma_1}\theta\,\frac{d\sigma}{r^3},\quad 4\pi R_1\theta_n=\left[l^2-\left(\frac{R_1}{R_2}\right)^{4n}R_1^2\right]\int_{\sigma_1}\theta\,\frac{d\sigma}{\mathbf{r}_{2n}^3},\\
&4\pi R_1\theta_{-n}=\left[\left(\frac{R_2}{R_1}\right)^{4n}R_1^2-l^2\right]\int_{\sigma_1}\theta\,\frac{d\sigma}{\mathbf{r}'^3_{2n-1}},
\end{aligned}\right\}\qquad(43)$$

$$\left.\begin{gathered}4\pi R_2\theta'_0 = (R_2^2 - l^2)\int_{\sigma_2}\theta\frac{d\sigma}{r^3}, \quad 4\pi R_2\theta'_n = \left[\left(\frac{R_2}{R_1}\right)^{4n}R_1^2 - l^2\right]\int_{\sigma_2}\theta\frac{d\sigma}{r'^3_{2n}},\\ 4\pi R_2\theta'_{-n} = \left[l^2 - \left(\frac{R_1}{R_2}\right)^{4n}R_2^2\right]\int_{\sigma_2}\theta\frac{d\sigma}{r^3_{2n-1}},\\ 4\pi R_1\varphi_0 = \int_{\sigma_1}\theta\frac{d\sigma}{r}, \quad 4\pi R_1\varphi_n = \int_{\sigma_1}\theta\frac{d\sigma}{r_{2n}}, \quad 4\pi R_1\varphi_{-n} = \int_{\sigma_1}\theta\frac{d\sigma}{r'_{2n-1}},\\ 4\pi R_2\varphi'_0 = \int_{\sigma_2}\theta\frac{d\sigma}{r}, \quad 4\pi R_2\varphi'_n = \int_{\sigma_2}\theta\frac{d\sigma}{r'_{2n}}, \quad 4\pi R_1\varphi'_{-n} = \int_{\sigma_2}\theta\frac{d\sigma}{r_{2n-1}}\end{gathered}\right\} \quad (43)$$

the usual formulas (5) and similar ones in report I will in the present case give us

$$\left.\begin{gathered}u = U - \frac{\lambda+\mu}{2\mu}x\theta + \frac{\lambda+\mu}{2\mu}\Big\{x\theta + (l^2 - R_1^2)\frac{\partial\varphi_0}{\partial x} +\\ + \sum_1^\infty\left(\frac{R_2}{R_1}\right)^n\left[x\theta_n + [l^2 - \left(\frac{R_1}{R_2}\right)^{4n}R_1^2]\frac{\partial\varphi_n}{\partial x}\right] -\\ - \sum_1^\infty\left(\frac{R_1}{R_2}\right)^n\left[x\theta_{-n} + [\left(\frac{R_2}{R_1}\right)^{4n}R_1^2 - l^2]\frac{\partial\varphi_{-n}}{\partial x}\right] +\\ + x\theta'_0 + (R_2^2 - l^2)\frac{\partial\varphi'_0}{\partial x} + \sum_1^\infty\left(\frac{R_1}{R_2}\right)^n\left[x\theta'_n + [\left(\frac{R_2}{R_1}\right)^n R_2^2 - l^2]\frac{\partial\varphi'_n}{\partial x}\right] -\\ - \sum_1^\infty\left(\frac{R_2}{R_1}\right)^n\left[x\theta'_{-n} + [l^2 - \left(\frac{R_1}{R_2}\right)^{4n}R_2^2]\frac{\partial\varphi'_{-n}}{\partial x}\right]\Big\} \;(*).\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{gathered}\right\} \quad (44)$$

The other two unwritten formulas are derived from the written one by exchanging u, U, x with v, V, y and with w, W, z. /48

The attempt must now be made to determine the unknown functions θ_0, θ_n, θ_{-n}; θ'_0, θ'_n, θ'_{-n}; ϕ_0, ..., so that: (1) the series which appear on the right sides of expressions (44) will be convergent in S and will tend on σ toward finite values; (2) each of these series will be harmonic and regular in S; (3) between θ with the different subscripts and the corresponding values of ϕ the following relationships hold true; (4) the ϕ and ϕ' with subscripts different from zero will be linked with $\phi_0(x, y, z)$ and $\phi'_0(x, y, z)$ by the following relationships:

$$\left.\begin{gathered}-\theta_0 = 2l\frac{\partial\varphi_0}{\partial l} + \varphi_0, \quad -\theta_n = 2l\frac{\partial\varphi_n}{\partial l} + \varphi_n, \quad \theta_{-n} = 2l\frac{\partial\varphi_{-n}}{\partial l} + \varphi_{-n},\\ \theta'_0 = 2l\frac{\partial\varphi'_0}{\partial l} + \varphi'_0, \quad \theta'_n = 2l\frac{\partial\varphi'_n}{\partial l} + \varphi'_n, \quad -\theta'_{-n} = 2l\frac{\partial\varphi'_{-n}}{\partial l} + \varphi'_{-n},\end{gathered}\right\} \quad (45')$$

* It is useful for the following discussion to bear in mind the fact that the quantity in brackets { } is an expression of the function which is harmonic and regular in S which assumes values xθ on σ.

$$
\left.\begin{aligned}
\varphi_n(x, y, z) &= \left(\frac{R_2}{R_1}\right)^{2n} \varphi_0\left[\left(\frac{R_2}{R_1}\right)^{2n} x, \ \left(\frac{R_2}{R_1}\right)^{2n} y, \ \left(\frac{R_2}{R_1}\right)^{2n} z\right], \\
\varphi_{-n} &= \frac{R_1}{l}\varphi_0\left[\left(\frac{R_2}{R_1}\right)^{2n}\frac{R_1^2 x}{l^2}, \left(\frac{R_2}{R_1}\right)^{2n}\frac{R_1^2 y}{l^2}, \left(\frac{R_2}{R_1}\right)^{2n}\frac{R_1^2 z}{l^2}\right], \\
\varphi'_n(x, y, z) &= \left(\frac{R_1}{R_2}\right)^{2n} \varphi'_0\left[\left(\frac{R_1}{R_2}\right)^{2n} x, \ \left(\frac{R_1}{R_2}\right)^{2n} y, \ \left(\frac{R_1}{R_2}\right)^{2n} z\right], \\
\varphi'_{-n} &= \frac{R_2}{l}\varphi'_0\left[\left(\frac{R_1}{R_2}\right)^{2n}\frac{R_2^2 x}{l^2}, \left(\frac{R_1}{R_2}\right)^{2n}\frac{R_2^2 y}{l^2}, \left(\frac{R_1}{R_2}\right)^{2n}\frac{R_2^2 z}{l^2}\right],
\end{aligned}\right\} \tag{45'}
$$

(5) finally, the following equation will be identically satisfied:

$$
\begin{aligned}
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} &= \frac{3\lambda + 5\mu}{2\mu}\theta + \frac{\lambda+\mu}{2\mu} l \frac{\partial \theta}{\partial l} - \\
-\frac{\lambda+\mu}{2\mu}\Bigg\{ 3\theta_0 + l\frac{\partial \theta_0}{\partial l} &+ 2l\frac{\partial \varphi_0}{\partial l} + \sum_1^\infty{}_n \left(\frac{R_2}{R_1}\right)^n \left(3\theta_n + l\frac{\partial \theta_n}{\partial l} + 2l\frac{\partial \varphi_n}{\partial l}\right) - \\
&- \sum_1^\infty{}_n \left(\frac{R_1}{R_2}\right)^n \left(3\theta_{-n} + l\frac{\partial \theta_{-n}}{\partial l} - 2l\frac{\partial \varphi_{-n}}{\partial l}\right) + \\
+ 3\theta'_0 + l\frac{\partial \theta'_0}{\partial l} - 2l\frac{\partial \varphi'_0}{\partial l} &+ \sum_1^\infty{}_n \left(\frac{R_1}{R_2}\right)^n \left(3\theta'_n + l\frac{\partial \theta'_n}{\partial l} - 2l\frac{\partial \varphi'_n}{\partial l}\right) - \\
&- \sum_1^\infty{}_n \left(\frac{R_2}{R_1}\right)^n \left(3\theta'_{-n} + l\frac{\partial \theta'_{-n}}{\partial l} + 2l\frac{\partial \varphi'_{-n}}{\partial l}\right)\Bigg\}.
\end{aligned}
$$

It is easily verified that under these conditions expressions (44) represent the solution of our problem.

To achieve our purpose, let us observe that, since ϕ_0 must be a function which is harmonic and regular outside the sphere of radius R_1, it may always be represented with a series of form /49

$$
\varphi_0 = \sum_0^\infty{}_m X_{-(m+1)}, \tag{47}
$$

Since ϕ'_0 must be harmonic and regular inside the sphere of radius R_2, it may be represented by a series of form

$$
\varphi'_0 = \sum_0^\infty{}_m X_m, \tag{47'}
$$

where $X_{-(m+1)}$ and X_m are solid spherical harmonic functions, the first of negative order $-(m + 1)$ and the second of positive order *m* to be determined in such a way that the preceding conditions are satisfied. We will then have

$$
\left.\begin{aligned}
\varphi_n &= \sum_0^\infty{}_m \left(\frac{R_1}{R_2}\right)^{2nm} X_{-(m+1)}, \quad \varphi_{-n} = \sum_0^\infty{}_m \left(\frac{R_1}{R_2}\right)^{2n(m+1)} \left(\frac{l}{R_1}\right)^{2m+1} X_{-(m+1)}, \\
\varphi'_n &= \sum_0^\infty{}_m \left(\frac{R_1}{R_2}\right)^{2n(m+1)} X_m, \quad \varphi'_{-n} = \sum_0^\infty{}_m \left(\frac{R_1}{R_2}\right)^{2nm} \left(\frac{R_2}{l}\right)^{2m+1} X_m; \\
\theta_0 &= \sum_0^\infty{}_m (2m+1) X_{-(m+1)}, \quad \theta_n = \sum_0^\infty{}_m (2m+1)\left(\frac{R_1}{R_2}\right)^{2nm} X_{-(m+1)}, \\
\theta_{-n} &= \sum_0^\infty{}_m (2m+1)\left(\frac{R_1}{R_2}\right)^{2n(m+1)} \left(\frac{l}{R_1}\right)^{2m+1} X_{-(m+1)};
\end{aligned}\right\} \tag{47''}
$$

$$\theta'_0 = \sum_{m=0}^{\infty} (2m+1) X_m, \quad \theta'_n = \sum_{m=0}^{\infty} (2m+1)\left(\frac{R_1}{R_2}\right)^{2n(m+1)} X_m,$$

$$\theta'_{-n} = \sum_{m=0}^{\infty} (2m+1)\left(\frac{R_1}{R_2}\right)^{2nm}\left(\frac{R_2}{l}\right)^{2m+1} X_m, \tag{47''}$$

Noting that

$$\frac{\partial}{\partial x}(l X_{-1}) = 0 \quad \text{or} \quad \frac{\partial X_{-1}}{\partial x} = -\frac{x}{l^2} X_{-1},$$

for which the terms which contain X_{-1} in $x\theta_n + l^2 \frac{\partial \phi_n}{\partial x}$ and $\frac{\partial \phi_{-n}}{\partial x}$ are zero, we have /50

$$\left.\begin{aligned}
u = U &- \frac{\lambda+\mu}{2\mu} x\theta + \frac{\lambda+\mu}{2\mu}\Bigg\{ x \frac{R_2^3 - l^3}{R_2^3 - R_1^3}\left(\frac{R_1}{l}\right)^2 X_{-1} + \\
&+ x \sum_{m=1}^{\infty} (2m+1) \frac{R_2^{2m-1} - l^{2m-1}}{R_2^{2m-1} - R_1^{2m-1}} X_{-(m+1)} + \\
&+ \sum_{m=1}^{\infty} \left[l^2 \frac{R_2^{2m-1} - l^{2m-1}}{R_2^{2m-1} - R_1^{2m-1}} - R_1^2 \frac{R_2^{2m+3} - l^{2m+3}}{R_2^{2m+3} - R_1^{2m+3}} \right] \frac{\partial X_{-(m+1)}}{\partial x} + \\
&+ x \sum_{m=0}^{\infty} (2m+1) \frac{l^{2m+3} - R_1^{2m+3}}{R_2^{2m+3} - R_1^{2m+3}} \left(\frac{R_2}{l}\right)^{2m+3} X_m + \\
&+ \sum_{m=1}^{\infty} \left[l^2 \frac{l^{2m-1} - R_1^{2m-1}}{R_2^{2m-1} - R_1^{2m-1}} - R_2^2 \frac{l^{2m+3} - R_1^{2m+3}}{R_2^{2m+3} - R_1^{2m+3}} \right] \left(\frac{R_2}{l}\right)^{2m+1} \frac{\partial X_m}{\partial x} \Bigg\}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\theta = &\sum_{m=0}^{\infty} (2m+1) \frac{R_2^{2m+1} - l^{2m+1}}{R_2^{2m+1} - R_1^{2m+1}} X_{-(m+1)} + \\
&+ \sum_{m=0}^{\infty} (2m+1) \frac{l^{2m+1} - R_1^{2m+1}}{R_2^{2m+1} - R_1^{2m+1}} \left(\frac{R_2}{l}\right)^{2m+1} X_m,
\end{aligned}\right\} \tag{48}$$

in which again the two formulas relative to v and w, which are not written, are deduced from the formula for u by interchanging u, U, x and v, V, y, and u, U, x and w, W, z.

We now assume

$$\left.\begin{aligned}
U &= \sum_{m=0}^{\infty} U_m + \sum_{m=0}^{\infty} U_{-(m+1)}, \\
V &= \sum_{m=0}^{\infty} V_m + \sum_{m=0}^{\infty} V_{-(m+1)}, \\
W &= \sum_{m=0}^{\infty} W_m + \sum_{m=0}^{\infty} W_{-(m+1)},
\end{aligned}\right\} \tag{49}$$

where U_m, $U_{-(m+1)}$, ... may be assumed to be known. Let us substitute into expression (46) the preceding values of U, V, W and the values of θ and ϕ given by expressions (47), (47'), and (47") and let us in expression (46) set the terms of degree m equal to those of degree $-(m + 1)$.

Assuming also

$$\Theta_m = \frac{\partial U_{m+1}}{\partial x} + \frac{\partial V_{m+1}}{\partial y} + \frac{\partial W_{m+1}}{\partial z}, \quad \Theta_{-(m+1)} = \frac{\partial U_{-m}}{\partial x} + \frac{\partial U_{-m}}{\partial y} + \frac{\partial W_{-m}}{\partial z},$$

we will have the equations /51

$$\begin{gathered}
\Theta_0 = X_0, \quad X_{-1} = \frac{R_1}{l} X_0, \\
2\mu\Theta_m = \\
= l^{2m+1} X_{-(m+1)} \left\{ \frac{(m+1)(2m+3)R_1^2(\lambda+\mu)}{R_2^{2m+3} - R_1^{2m+3}} - \frac{(2m+1)[(m+3)\lambda + (m+5)\mu]}{R_2^{2m+1} - R_1^{2m+1}} \right\} - \\
- R_2^{2m+1} X_m \left\{ \frac{(m+1)(2m+3)R_2^2(\lambda+\mu)}{R_2^{2m+3} - R_1^{2m+3}} - \frac{(2m+1)[(m+3)\lambda + (m+5)\mu]}{R_2^{2m+1} - R_1^{2m+1}} \right\}, \\
2\mu \left(\frac{l}{R_2} \right)^{2m+1} \Theta_{-(m+1)} = \\
= l^{2m+1} X_{-(m+1)} \left\{ \frac{m(2m-1)(\lambda+\mu)}{R_2^2(R_2^{2m-1} - R_1^{2m-1})} - \frac{(2m+1)[(m-2)\lambda + (m-4)\mu]}{R_2^{2m+1} - R_1^{2m+1}} \right\} - \\
- R_1^{2m+1} X_m \left\{ \frac{m(2m-1)(\lambda+\mu)}{R_1^2(R_2^{2m-1} - R_1^{2m-1})} - \frac{(2m+1)[(m-2)\lambda + (m-4)\mu]}{R_2^{2m+1} - R_1^{2m+1}} \right\},
\end{gathered} \tag{50}$$

in which subscript m ranges from 1 to ∞. Solving the preceding equations for $X_{-(m+1)}$ and X_m, we find

$$\begin{gathered}
X_0 = \Theta_0, \quad X_{-1} = \frac{R_1}{l} \Theta_0 \\
\Delta X_{-(m+1)} = \\
= 2\mu \left\{ \frac{m(2m-1)(\lambda+\mu)}{R_1^2(R_2^{2m-1} - R_1^{2m-1})} - \frac{(2m+1)[(m-2)\lambda + (m-4)\mu]}{R_2^{2m+1} - R_1^{2m+1}} \right\} \left(\frac{R_1}{l} \right)^{2m+1} \Theta_m - \\
- 2\mu \left\{ \frac{(m+1)(2m+3)R_2^2(\lambda+\mu)}{R_2^{2m+3} - R_1^{2m+3}} - \frac{(2m+1)[(m+3)\lambda + (m+5)\mu]}{R_2^{2m+1} - R_1^{2m+1}} \right\} \Theta_{-(m+1)}, \\
\Delta X_m = 2\mu \left\{ \frac{m(2m-1)(\lambda+\mu)}{R_2^2(R_2^{2m-1} - R_1^{2m-1})} - \frac{(2m+1)[(m-2)\lambda + (m-4)\mu]}{R_2^{2m+1} - R_1^{2m+1}} \right\} \Theta_m - \\
- 2\mu \left\{ \frac{(m+1)(2m+3)R_1^2(\lambda+\mu)}{R_2^{2m+3} - R_1^{2m+3}} - \frac{(2m+1)[(m+3)\lambda + (m+5)\mu]}{R_2^{2m+1} - R_1^{2m+1}} \right\} \left(\frac{l}{R_2} \right)^{2m+1} \Theta_{-(m+1)}, \\
\Delta = (R_2^{2m+1} - R_1^{2m+1}) \left\{ - \frac{m(m+1)(2m-1)(2m+3)(\lambda+\mu)^2}{(R_2^{2m+3} - R_1^{2m+3})(R_2^{2m-1} - R_1^{2m-1})} + \right. \\
\left. + \frac{(2m+1)^2[m(m+1)(\lambda+\mu)^2 + 4\mu(\lambda+2\mu)]}{(R_2^{2m+1} - R_1^{2m+1})^2} \right\}.
\end{gathered} \tag{51}$$

Since $\mu(\lambda + 2\mu) > 0$, Δ never becomes zero for any whole and positive value of m or for any value of R_1 such that $0 < R_1 < R_2$.* This shows that the /52

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* It is somewhat difficult to prove this assertion. The goal may be reached by the following line of reasoning. (Continued on following page).

problem always has one solution and one only. To convince ourselves of the convergence of the series which appear in our formulas in region S, it is sufficient to compare them to the series

$$\sum_{0}^{\infty}{}_m \left(\frac{R_1}{l}\right)^{2m+1}\Theta_m, \quad \sum_{0}^{\infty}{}_m \Theta_{-(m+1)}$$

which under the assumptions converge in absolute and uniform fashion between $l > R_1$, or to the series /53

$$\sum_{0}^{\infty}{}_m \Theta_m, \quad \sum_{0}^{\infty}{}_m \left(\frac{l}{R_2}\right)^{2m+1}\Theta_{m(m+1)}$$

which converge absolutely and uniformly for $l < R_2$.

This comparison may be made without difficulty. Finally, the various

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* (Continued)

To show that Δ never becomes zero under the conditions indicated, it is sufficient to demonstrate the fact that this property is possessed by the expression

$$\left.\begin{aligned} &-(R_2^{2m+1}-R_1^{2m+1})^2\, m(m+1)(2m-1)(2m+3)(\lambda+\mu)^2+\\ &+(2m+1)^2[m(m+1)(\lambda+\mu)^2+4\mu(\lambda+2\mu)]\,(R_2^{2m+3}-R_1^{2m+3})(R_2^{2m-1}-R_1^{2m-1}).\end{aligned}\right\} \quad \text{(a)}$$

Expression (a) for $R_1 = R_2$ becomes zero, while for $R_1 = 0$ it is greater than zero. It is therefore sufficient to show that expression (a) decreases continuously as R_1 increases from zero to R_2, or that the derivative with respect to R_1 is always negative, whatever m may be, and for $0 < R_1 < R_2$. This derivative is given by

$$\left.\begin{aligned} &(2m+1)R_1^{2m-2}\{2R_1^2(R_2^{2m+1}-R_1^{2m+1})\,m(m+1)(2m-1)(2m+3)(\lambda+\mu)^2-\\ &\qquad-(2m+1)[m(m+1)(\lambda+\mu)^2+\\ &+4\mu(\lambda+2\mu)][(2m+3)R_1^4(R_2^{2m-1}-R_1^{2m-1})+(2m-1)(R_2^{2m+3}-R_1^{2m+3})\}.\end{aligned}\right\} \quad \text{(b)}$$

The quantity in brackets { }, when we set $R_2 = R_1 e^{\tilde{\omega}}$, except for R_1^{2m+3}, may be written

$$\begin{aligned} &2(e^{(2m+1)\tilde{\omega}}-1)\,m(m+1)(2m-1)(2m+3)(\lambda+\mu)^2-(2m+1)[m(m+1)(\lambda+\mu)^2+\\ &\qquad+4\mu(\lambda+2\mu)][(2m+3)(e^{(2m-1)\tilde{\omega}}-1)+(2m-1)(e^{(2m+3)\tilde{\omega}}-1)]\end{aligned}$$

and expanding the exponentials in series, except for factor (2m - 1) (2m + 1) (2m + 3), we may also write

$$\begin{aligned} &-\sum_{1}^{\infty}{}_\nu \frac{\tilde{\omega}^\nu}{\underline{|\nu}}\{m(m+1)(\lambda+\mu)^2[(2m+3)^{\nu-1}+(2m-1)^{\nu-1}-2(2m+1)^{\nu-1}]+\\ &\qquad+4\mu(\lambda+2\mu)[(2m+3)^{\nu-1}+(2m-1)^{\nu-1}]\}.\end{aligned}$$

Since

$$\begin{aligned} &(2m+3)^{\nu-1}+(2m-1)^{\nu-1}-2(2m+1)^{\nu-1}=\sum_{0}^{\nu-1}{}_i \binom{\nu-1}{i}(2m)^i\,[3^{\nu-1-i}+\\ &\qquad+(-1)^{\nu-1-i}-2]>0,\end{aligned}$$

this formula completely proves our assertion.

terms on the right sides of expressions (44) and (48) will converge on σ to finite limits if the second derivatives of series (49) possess this property, which may certainly be brought about by assuming that the values of u, v, w given on σ admit derivatives, of the first order at least, with respect to the two parameters which specify the points in σ.

5. *Case in Which L, M, N are Given on the Two Limiting Spheres.* Let us start from the equations

$$\left.\begin{aligned} &\Delta^2\left(l\frac{\partial u}{\partial l}\right) + \frac{\lambda+\mu}{\mu}\frac{\partial}{\partial x}\left(l\frac{\partial \theta}{\partial l} + \theta\right) = 0, \dots \\ &l\frac{\partial \theta}{\partial l} + \theta = \frac{\partial}{\partial x}\left(l\frac{\partial u}{\partial l}\right) + \frac{\partial}{\partial y}\left(l\frac{\partial v}{\partial l}\right) + \frac{\partial}{\partial z}\left(l\frac{\partial w}{\partial l}\right) \end{aligned}\right\} \tag{52}$$

and write the usual basic formulas for these equations, still retaining for a moment Green's function G

$$\left.\begin{aligned} l\frac{\partial u}{\partial l} = {} & \frac{R_1}{4\pi}\int_{\sigma_1}\frac{\partial u}{\partial \rho}\frac{dG}{dn}d\sigma + \frac{R_2}{4\pi}\int_{\sigma_2}\frac{\partial u}{\partial \rho}\frac{dG}{dn}d\sigma - \frac{\lambda+\mu}{2\mu}x\left(l\frac{\partial \theta}{\partial l} + \theta\right) + \\ & + \frac{\lambda+\mu}{8\pi\mu}\int_{\sigma}\xi\left(\rho\frac{\partial \theta}{\partial \rho} + \theta\right)\frac{dG}{dn}d\sigma, \\ & \dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{53}$$

We should remember that the surface conditions are now

$$\left.\begin{aligned} &\text{on } \sigma_1 \qquad -L = \lambda\theta\frac{x}{R_1} + 2\mu\left(\frac{\partial u}{\partial l} + \varpi_3\frac{y}{R_1} - \varpi_2\frac{z}{R_1}\right), \dots \\ &\text{on } \sigma_2 \qquad L = \lambda\theta\frac{x}{R_2} + 2\mu\left(\frac{\partial u}{\partial l} + \varpi_3\frac{y}{R_2} - \varpi_2\frac{z}{R_2}\right), \dots \end{aligned}\right\} \tag{54}$$

and we shall call $\mathfrak{L}$; $\mathfrak{M}$, $\mathfrak{N}$ the functions which are harmonic and regular in S, which on σ_1 assume the values of L, M, N given on this surface and multiplied by $-R_1$, and on σ_2 the values of L, M, N given on σ_2 and multiplied by R_2. Expressions (53) may then be written /54

$$\left.\begin{aligned} l\frac{\partial u}{\partial l} = {} & \frac{\mathfrak{L}}{2\mu} - \frac{1}{8\pi\mu}\int_{\sigma}[\lambda\theta\xi + 2\mu(\varpi_3\eta - \varpi_2\zeta)]\frac{dG}{dn}d\sigma - \\ & - \frac{\lambda+\mu}{2\mu}x\left(l\frac{\partial \theta}{\partial l} + \theta\right) + \frac{\lambda+\mu}{8\pi\mu}\int_{\sigma}\xi\left(\rho\frac{\partial \theta}{\partial \rho} + \theta\right)\frac{dG}{dn}d\sigma, \\ & \dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{53'}$$

and it is a question first of determining θ; $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}_3$ so that the equations

$$\left.\begin{aligned} &l\frac{\partial \theta}{\partial l} + \theta = \frac{\partial}{\partial x}\left(l\frac{\partial u}{\partial l}\right) + \frac{\partial}{\partial y}\left(l\frac{\partial v}{\partial l}\right) + \frac{\partial}{\partial z}\left(l\frac{\partial w}{\partial l}\right); \\ &2\left(l\frac{\partial \varpi_1}{\partial l} + \varpi_1\right) = \frac{\partial}{\partial y}\left(l\frac{\partial w}{\partial l}\right) - \frac{\partial}{\partial z}\left(l\frac{\partial v}{\partial l}\right), \dots \end{aligned}\right\} \tag{55}$$

are identically satisfied.

Therefore we set

$$
\left.\begin{aligned}
\theta &= \sum_0^\infty{}_m (2m+1) \frac{R_2^{2m+1} - l^{2m+1}}{R_2^{2m+1} - R_1^{2m+1}} X_{-(m+1)} + \\
&+ \sum_0^\infty{}_m (2m+1) \frac{l^{2m+1} - R_1^{2m+1}}{R_2^{2m+1} - R_1^{2m+1}} \left(\frac{R_2}{l}\right)^{2m+1} X_m, \\
\varpi_i &= \sum_0^\infty{}_m (2m+1) \frac{R_2^{2m+1} - l^{2m+1}}{R_2^{2m+1} - R_1^{2m+1}} Y_{i,-(m+1)} + \\
&+ \sum_0^\infty{}_m (2m+1) \frac{l^{2m+1} - R_1^{2m+1}}{R_2^{2m+1} - R_1^{2m+1}} \left(\frac{R_2}{l}\right)^{2m+1} Y_{i,m} \qquad i = 1,\ 2,\ 3.
\end{aligned}\right\} \tag{56}
$$

Based on the results of the preceding section, we will then have

$$
\begin{aligned}
&\frac{1}{4\pi}\int_0 [\lambda \theta \xi + 2\mu(\varpi_3 \eta - \varpi_2 \zeta)] \frac{dG}{dn} d\sigma = \\
&= \frac{R_2^3 - l^3}{R_2^3 - R_1^3}\left(\frac{R_1}{l}\right)^2 [\lambda x X_{-1} + 2\mu(y Y_{3,-1} - z Y_{2,-1})] + \\
&+ \sum_1^\infty{}_m (2m+1) \frac{R_2^{2m-1} - l^{2m-1}}{R_2^{2m-1} - R_1^{2m-1}} [\lambda x X_{-(m+1)} + 2\mu(y Y_{3,-(m+1)} - z Y_{2,-(m+1)})] + \\
&+ \sum_1^\infty{}_m \left[l^2 \frac{R_2^{2m-1} - l^{2m-1}}{R_2^{2m-1} - R_1^{2m-1}} - R_1^2 \frac{R_2^{2m+3} - l^{2m+3}}{R_2^{2m+3} - R_1^{2m+3}}\right]\left[\lambda \frac{\partial X_{-(m+1)}}{\partial x} + \right. \\
&\left. + 2\mu\left(\frac{\partial Y_{3,-(m+1)}}{\partial y} - \frac{\partial Y_{2,-(m+1)}}{\partial z}\right)\right] +
\end{aligned} \tag{57}
$$

/55

$$
\begin{aligned}
&+ \sum_0^\infty{}_m (2m+1) \frac{l^{2m+3} - R_1^{2m+3}}{R_2^{2m+3} - R_1^{2m+3}} \left(\frac{R_2}{l}\right)^{2m+3} [\lambda x X_m + 2\mu(y Y_{3,m} - z Y_{2,m})] + \\
&+ \sum_1^\infty{}_m \left[l^2 \frac{l^{2m-1} - R_1^{2m-1}}{R_2^{2m-1} - R_1^{2m-1}} - R_2^2 \frac{l^{2m+3} - R_1^{2m+3}}{R_2^{2m+3} - R_1^{2m+3}}\right]\left(\frac{R_2}{l}\right)^{2m+1}\left[\lambda \frac{\partial X_m}{\partial x} + \right. \\
&\left. + 2\mu\left(\frac{\partial Y_{3,m}}{\partial y} - \frac{\partial Y_{2,m}}{\partial z}\right)\right].
\end{aligned}
$$

Let us then remember that, since

$$
(\lambda + 2\mu)\frac{\partial \theta}{\partial x} + 2\mu\left(\frac{\partial \varpi_2}{\partial z} - \frac{\partial \varpi_3}{\partial y}\right) = 0, \ldots
$$

must be true, the following relationships will hold:

$$
\begin{aligned}
&2\mu R_1 (z Y_{2,0} - y Y_{3,0}) - 2\mu l (z Y_{2,-1} - y Y_{3,-1}) = \\
&= (\lambda + 2\mu) x (l X_{-1} - R_1 X_0); \quad \frac{\partial Y_{i,-1}}{\partial x} = -\frac{x}{l^2} Y_{i,-1}, \quad i = 1, 2, 3 \\
&(R_2^{2m+1} - R_1^{2m+1})\left[\lambda \frac{\partial X_{-(m+1)}}{\partial x} + 2\mu\left(\frac{\partial Y_{3,-(m+1)}}{\partial y} - \frac{\partial Y_{2,-(m+1)}}{\partial z}\right)\right] = \\
&= 2(\lambda + \mu)\left[(R_2^{2m+1} - R_1^{2m+1}) \frac{\partial X_{-(m+1)}}{\partial x} + \right.
\end{aligned} \tag{58}
$$

$$+(2m+1)\left(\frac{R_1}{l}\right)^{2m+1}\frac{x}{l^2}\left(R_2^{2m+1}X_m - l^{2m+1}X_{-(m+1)}\right)\Big] -$$

$$-(2m+1)\frac{(R_1 R_2)^{2m+1}}{l^{2m+3}}\left[\lambda x X_m + 2\mu\left(y Y_{3,m} - z Y_{2,m}\right)\right] +$$

$$+(2m+1)\frac{R_1^{2m+1}}{l^2}\left[\lambda x X_{-(m+1)} + 2\mu\left(y Y_{3,-(m+1)} - z Y_{2,-(m+1)}\right)\right],$$

$$\left(R_2^{2m+1} - R_1^{2m+1}\right)\left[\lambda\frac{\partial X_m}{\partial x} + 2\mu\left(\frac{\partial Y_{3,m}}{\partial y} - \frac{\partial Y_{2,m}}{\partial z}\right)\right] = \qquad (58)$$

$$= 2(\lambda+\mu)\left[\left(R_2^{2m+1} - R_1^{2m+1}\right)\frac{\partial X_m}{\partial x} +\right.$$

$$\left. + (2m+1)\frac{x}{l^2}\left(R_1^{2m+1}X_m - l^{2m+1}X_{-(m+1)}\right)\right] -$$

$$-(2m+1)\frac{R_1^{2m+1}}{l^2}\left[\lambda x X_m + 2\mu\left(y Y_{3,m} - z Y_{2,m}\right)\right] +$$

$$+(2m+1)\,l^{2m-1}\left[\lambda x X_{-(m+1)} + 2\mu\left(y Y_{3,-(m+1)} - z Y_{2,-(m+1)}\right)\right],$$

$$m = 1, 2, \ldots, \infty$$

and the others which are derived from these by cyclic permutations of x, y, z /56
and the subscripts 1, 2, 3. By means of these relationships, expression (57) may be readily written in the form

$$\frac{1}{4\pi}\int_\sigma\left[\lambda\theta\xi + 2\mu\left(\varpi_3\eta - \varpi_2\zeta\right)\right]\frac{dG}{dn}\,d\sigma = \lambda\theta x + 2\mu\left(\varpi_3 y - \varpi_2 z\right) +$$

$$+2(\lambda+\mu)\frac{x}{R_1}\left[\frac{l^3 - R_1^3}{R_2^3 - R_1^3}\left(\frac{R_2}{l}\right)^2 - \frac{l - R_1}{R_2 - R_1}\right]\frac{R_2}{l}\left(-l X_{-1} + R_1 X_0\right) +$$

$$+2(\lambda+\mu)\sum_{1}^{\infty}{}_m\frac{1}{R_2^{2m+1} - R_1^{2m+1}}\left[l^2\frac{R_2^{2m-1} - l^{2m-1}}{R_2^{2m-1} - R_1^{2m-1}} -\right.$$

$$\left. - R_1^2\frac{R_2^{2m+3} - l^{2m+3}}{R_2^{2m+3} - R_1^{2m+3}}\right]\left[\left(R_2^{2m+1} - R_1^{2m+1}\right)\frac{\partial X_{-(m+1)}}{\partial x} +\right. \qquad (57')$$

$$\left. + (2m+1)\,x\frac{R_1^{2m+1}}{l^{2m+3}}\left(R_2^{2m+1}X_m - l^{2m+1}X_{-(m+1)}\right)\right] +$$

$$+2(\lambda+\mu)\sum_{1}^{\infty}{}_m\frac{R_2^{2m+1}}{\left(R_2^{2m+1} - R_1^{2m+1}\right)l^{2m+1}}\left[l^2\frac{l^{2m-1} - R_1^{2m-1}}{R_2^{2m-1} - R_1^{2m-1}} -\right.$$

$$\left. - R_2^2\frac{l^{2m+3} - R_1^{2m+3}}{R_2^{2m+3} - R_1^{2m+3}}\right]\left[\left(R_2^{2m+1} - R_1^{2m+1}\right)\frac{\partial X_m}{\partial x} +\right.$$

$$\left. + (2m+1)\frac{x}{l^2}\left(R_1^{2m+1}X_m - l^{2m+1}X_{-(m+1)}\right)\right],$$

and the other two similar relationships are derived by cyclic permutation of ξ, η, ζ; x, y, z; $\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3$.

The last terms in expression (53') are also easily formulated when we note that from the identity

$$
\left.\begin{aligned}
l\frac{\partial \theta}{\partial l} + \theta = -\sum_0^\infty{}_m (2m+1)\frac{m R_2^{2m+1} + (m+1) l^{2m+1}}{R_2^{2m+1} - R_1^{2m+1}} X_{-(m+1)} + \\
+ \sum_0^\infty{}_m (2m+1)\frac{(m+1) l^{2m+1} + m R_1^{2m+1}}{R_2^{2m+1} - R_1^{2m+1}}\left(\frac{R_2}{l}\right)^{2m+1} X_m = \\
= \sum_0^\infty{}_m (2m+1)\frac{R_2^{2m+1} - l^{2m+1}}{R_2^{2m+1} - R_1^{2m+1}} X'_{-(m+1)} + \\
+ \sum_0^\infty{}_m (2m+1)\frac{l^{2m+1} - R_1^{2m+1}}{R_2^{2m+1} - R_1^{2m+1}}\left(\frac{R_2}{l}\right)^{2m+1} X'_m ,
\end{aligned}\right\} \qquad (59)
$$

we obtain /57

$$
\left.\begin{aligned}
(R_2^{2m+1} - R_1^{2m+1}) X'_{-(m+1)} = \\
= -[m R_2^{2m+1} + (m+1) R_1^{2m+1}] X_{-(m+1)} + (2m+1)\left(\frac{R_1 R_2}{l}\right)^{2m+1} X_m , \\
(R_2^{2m+1} - R_1^{2m+1}) X'_m = -(2m+1) l^{2m+1} X_{-(m+1)} + \\
+ [m R_1^{2m+1} + (m+1) R_2^{2m+1}] X_m .
\end{aligned}\right\} \qquad (59')
$$

Therefore, applying the results of the preceding paragraph to the integrals:

$$
\frac{1}{4\pi}\int_\sigma \xi\left(\rho\frac{\partial \theta}{\partial \rho} + \theta\right)\frac{dG}{dn}\, d\sigma, \ldots
$$

and substituting into (53) the expressions relative to (57') and similar ones, we find

$$
\left.\begin{aligned}
l\frac{\partial u}{\partial l} = \frac{\varrho}{2\mu} - \frac{\lambda}{2\mu}\theta x - \varpi_3 y + \varpi_2 z - \frac{\lambda+\mu}{2\mu} x\left(l\frac{\partial \theta}{\partial l} + \theta\right) + \\
+ \frac{\lambda+\mu}{2\mu} x \frac{-l X_{-1} + R_2 X_0}{R_2 - R_1} - \frac{\lambda+\mu}{\mu}\frac{x}{R_1}\left[\frac{l^3 - R_1^3}{R_2^3 - R_1^3}\left(\frac{R_2}{l}\right)^2 - \right. \\
\left. - \frac{l - R_1}{R_2 - R_1}\right]\frac{R_2}{l}(-l X_{-1} + R_1 X_0) + \\
+ \frac{\lambda+\mu}{2\mu} x \sum_1^\infty{}_m (2m+1)\left[\frac{R_2^{2m-1} - l^{2m-1}}{R_2^{2m-1} - R_1^{2m-1}} X'_{-(m+1)} + \right. \\
\left. + \frac{l^{2m+3} - R_1^{2m+3}}{R_2^{2m+3} - R_1^{2m+3}}\left(\frac{R_2}{l}\right)^{2m+3} X'_m\right] + \\
+ \frac{\lambda+\mu}{2\mu}\sum_1^\infty{}_m\left[l^2\frac{R_2^{2m-1} - l^{2m-1}}{R_2^{2m-1} - R_1^{2m-1}} - R_1^2\frac{R_2^{2m+3} - l^{2m+3}}{R_2^{2m+3} - R_1^{2m+3}}\right]\left[\frac{\partial X'_{-(m+1)}}{\partial x} - \right. \\
\left. - 2\frac{\partial X_{-(m+1)}}{\partial x} - 2\frac{(2m+1) R_1^{2m+1} x}{(R_2^{2m+1} - R_1^{2m+1}) l^{2m+3}}(R_2^{2m+1} X_m - l^{2m+1} X_{-(m+1)})\right] + \\
+ \frac{\lambda+\mu}{2\mu}\sum_1^\infty{}_m\left[l^2\frac{l^{2m-1} - R_1^{2m-1}}{R_2^{2m-1} - R_1^{2m-1}} - R_2^2\frac{l^{2m+3} - R_1^{2m+3}}{R_2^{2m+3} - R_1^{2m+3}}\right]\left(\frac{R_2}{l}\right)^{2m+1}\left[\frac{\partial X'_m}{\partial x} - \right. \\
\left. - 2\frac{\partial X_m}{\partial x} - 2\frac{(2m+1) x}{(R_2^{2m+1} - R_1^{2m+1}) l^2}(R_1^{2m+1} X_m - l^{2m+1} X_{-(m+1)})\right],
\end{aligned}\right\} \qquad (53'')
$$

. . . .

The other two formulas for v and w are deduced from that relative to u by cyclical interchange of $\mathfrak{L}, \mathfrak{M}, \mathfrak{N}$; $\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3$; x, y, z.

Now substituting expression (53") into the first of expressions (55) and utilizing the quoted relationships between θ; $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}_3$, we find: /58

$$
\left.
\begin{aligned}
&\frac{\partial \mathfrak{L}}{\partial x}+\frac{\partial \mathfrak{M}}{\partial y}+\frac{\partial \mathfrak{N}}{\partial z}=(\lambda+\mu)\, l\,\frac{\partial}{\partial l}\left(l\frac{\partial \theta}{\partial l}+\theta\right)+\\
&+3(\lambda+\mu)\left(l\frac{\partial \theta}{\partial l}+\theta\right)+(3\lambda+2\mu)\theta+\\
&+(\lambda+\mu)\left[\frac{6R_2^2}{R_2^3-R_1^3}-\frac{9l-4R_1}{(R_2-R_1)\,l}\right]R_2X_0-\\
&-(\lambda+\mu)\left[\frac{6R_2^3}{R_2^3-R_1^3}-\frac{3(2R_2+R_1)l-4R_1R_2}{(R_2-R_1)\,l}\right]\frac{l}{R_1}X_{-1}+\\
&+(\lambda+\mu)\sum_1^\infty{}_m\left\{\left[\frac{m(2m-1)R_2^{2m-1}}{R_2^{2m-1}-R_1^{2m-1}}+\frac{(m+1)(2m+3)R_2^2}{R_2^{2m+3}-R_1^{2m+3}}\,l^{2m+1}\right]X'_{-(m+1)}-\right.\\
&\left.-\left[\frac{m(2m-1)R_1^{2m-1}}{(R_2^{2m-1}-R_1^{2m-1})\,l^{2m+1}}+\frac{(m+1)(2m+3)R_2^2}{R_2^{2m+3}-R_1^{2m+3}}\right]R_2^{2m+1}X'_m\right\}-\\
&-2(\lambda+\mu)\sum_1^\infty{}_m\frac{X_{-(m+1)}}{R_2^{2m+1}-R_1^{2m+1}}\left\{R_2^{2m-1}\left[2(m+1)R_2^2+\right.\right.\\
&\left.\left.+\frac{m(2m-1)R_1^{2m-1}(R_2^2-R_1^2)}{R_2^{2m-1}-R_1^{2m-1}}\right]-l^{2m+1}\left[2m+\frac{(m+1)(2m+3)R_2^{2m+1}(R_2^2-R_1^2)}{R_2^{2m+3}-R_1^{2m+3}}\right]\right\}+\\
&+2(\lambda+\mu)\sum_1^\infty{}_m\frac{X_m}{R_2^{2m+1}-R_1^{2m+1}}\left(\frac{R_2}{l}\right)^{2m+1}\left\{R_1^{2m-1}\left[2(m+1)R_1^2+\right.\right.\\
&\left.\left.+\frac{m(2m-1)R_2^{2m-1}(R_2^2-R_1^2)}{R_2^{2m-1}-R_1^{2m-1}}\right]-l^{2m+1}\left[2m+\frac{(m+1)(2m+3)R_1^{2m+1}(R_2^2-R_1^2)}{R_2^{2m+3}-R_1^{2m+3}}\right]\right\}.
\end{aligned}
\right\} \qquad (60)
$$

If then we set

$$
\left.
\begin{aligned}
&\mathfrak{L}=\sum_0^\infty{}_m\mathfrak{L}_m+\sum_1^\infty{}_m\mathfrak{L}_{-(m+1)},\quad \mathfrak{M}=\sum_1^\infty{}_m\mathfrak{M}_m+\sum_0^\infty{}_m\mathfrak{M}_{-(m+1)},\\
&\mathfrak{N}=\sum_0^\infty{}_m\mathfrak{N}_m+\sum_0^\infty{}_m\mathfrak{N}_{-(m+1)},\\
&\Theta_m=\frac{\partial \mathfrak{L}_{m+1}}{\partial x}+\frac{\partial \mathfrak{M}_{m+1}}{\partial y}+\frac{\partial \mathfrak{N}_{m+1}}{\partial z},\\
&\Theta_{-(m+1)}=\frac{\partial \mathfrak{L}_{-m}}{\partial x}+\frac{\partial \mathfrak{M}_{-m}}{\partial y}+\frac{\partial \mathfrak{N}_{-m}}{\partial z}
\end{aligned}
\right\} \qquad (61)
$$

and in equation (60) set the terms of degree m equal to those of degree $-(m+1)$ we obtain /59

$$
\begin{aligned}
&\Theta_0=(6\lambda+5\mu)\frac{-lX_{-1}+R_2X_0}{R_2-R_1}+\\
&+3(\lambda+\mu)\left(\frac{2R_2^2}{R_2^3-R_1^3}-\frac{3}{R_2-R_1}\right)R_2X_0-\\
&-3(\lambda+\mu)\left(\frac{2R_2^3}{R_2^3-R_1^3}-\frac{2R_2+R_1}{R_2-R_1}\right)\frac{l}{R_1}X_{-1},\\
&0=(\lambda+2\mu)\frac{R_2}{(R_2-R_1)\,l}(R_1X_0-lX_{-1}),
\end{aligned}
\qquad (62)
$$

$$\left.\begin{aligned}
&(R_2^{2m+1}-R_1^{2m+1})\,\Theta_m=-[2m(m-1)(\lambda+\mu)+\\
&+(2m+1)(3\lambda+2\mu)]\,(l^{2m+1}X_{-(m+1)}-R_2^{2m+1}X_m)+\\
&+(\lambda+\mu)\frac{(m+1)(m+2)(2m+3)R_2^{2m+1}(R_2^2-R_1^2)}{R_2^{2m+3}-R_1^{2m+3}}(l^{2m+1}X_{-(m+1)}-R_1^{2m+1}X_m),\\
&(R_2^{2m+1}-R_1^{2m+1})\left(\frac{l}{R_2}\right)^{2m+1}\Theta_{-(m+1)}=[-2(m+1)(m+2)(\lambda+\mu)+\\
&+(2m+1)(3\lambda+2\mu)]\,(l^{2m+1}X_{-(m+1)}-R_1^{2m+1}X_m)+\\
&+(\lambda+\mu)\frac{m(m-1)(2m-1)R_1^{2m-1}(R_2^2-R_1^2)}{(R_2^{2m-1}-R_1^{2m-1})R_2^2}(l^{2m+1}X_{-(m+1)}-R_2^{2m+1}X_m).
\end{aligned}\right\}\qquad(62)$$

The second of these equations requires that

$$R_1X_0-lX_{-1}=0$$

and then from the first equation (58) and similar ones, we also find that

$$R_1Y_{i,0}-lY_{i,-1}=0,\quad i=1,2,3.$$

This result shows that the terms of degree -1 in θ; $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}_3$ are identically zero.

Moreover, the first expression (62) gives us

$$X_0=\frac{\Theta_0}{3\lambda+2\mu}.\qquad(63)$$

Finally, the last two equations (62) give us /60

$$\left.\begin{aligned}
\Delta X_m=&\Big[2(m+1)(m+2)(\lambda+\mu)-(2m+1)(3\lambda+2\mu)-\\
&-(\lambda+\mu)\frac{m(m-1)(2m-1)R_1^{2m-1}(R_2^2-R_1^2)}{(R_2^{2m-1}-R_1^{2m-1})R_2^2}\Big]\Theta_m-\\
&-\Big[2m(m-1)(\lambda+\mu)+(2m+1)(3\lambda+2\mu)-\\
&-(\lambda+\mu)\frac{(m+1)(m+2)(2m+3)R_2^{2m+1}(R_2^2-R_1^2)}{R_2^{2m+3}-R_1^{2m+3}}\Big]\left(\frac{l}{R_2}\right)^{2m+1}\Theta_{-(m+1)},\\
\Delta X_{-(m+1)}=&\Big[2(m+1)(m+2)(\lambda+\mu)-(2m+1)(3\lambda+2\mu)-\\
&-(\lambda+\mu)\frac{m(m-1)(2m-1)R_2^{2m-1}(R_2^2-R_1^2)}{(R_2^{2m-1}-R_1^{2m-1})R_1^2}\Big]\left(\frac{R_1}{l}\right)^{2m+1}\Theta_m-\\
&-\Big[2m(m-1)(\lambda+\mu)+(2m+1)(3\lambda+2\mu)-\\
&-(\lambda+\mu)\frac{(m+1)(m+2)(2m+3)R_1^{2m+1}(R_2^2-R_1^2)}{R_2^{2m+3}-R_1^{2m+3}}\Big]\Theta_{-(m+1)},
\end{aligned}\right\}\qquad(64)$$

where

$$\left.\begin{aligned}
&\Delta=(2m+1)^2(3\lambda+2\mu)(\lambda+2\mu)+\\
&+m(m-1)(m+1)(m+2)(\lambda+\mu)^2\left[4-\frac{(2m-1)(2m+3)(R_1R_2)^{2m-1}(R_2^2-R_1^2)^2}{(R_2^{2m+3}-R_1^{2m+3})(R_2^{2m-1}-R_1^{2m-1})}\right].
\end{aligned}\right\}\qquad(64')$$

Just as in the preceding case, it is shown that Δ differs from zero for any whole and positive value of m since $3\lambda + 2\mu > 0$, $\mu > 0$. It is clearly sufficient to restrict ourselves to demonstrating the fact that expression

$$4 - \frac{(2m-1)(2m+3)(R_1 R_2)^{2m-1}(R_2^2 - R_1^2)^2}{(R_2^{2m+3} - R_1^{2m+3})(R_2^{2m-1} - R_1^{2m-1})}$$

possesses this property

When X_m and $X_{-(m+1)}$ are thus determined, θ is found from the first of expressions (56).

To find $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}_3$, expressions (53") are substituted into the last relationships (55). Bearing in mind relationship $\frac{\partial \bar{\omega}_1}{\partial x} + \frac{\partial \bar{\omega}_2}{\partial y} + \frac{\partial \bar{\omega}_3}{\partial z} = 0$, we meanwhile find /61

$$\begin{aligned}2\mu l \frac{\partial \bar{\omega}_1}{\partial l} &= \frac{\partial \mathfrak{N}}{\partial y} - \frac{\partial \mathfrak{M}}{\partial z} + \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\Big\{ \lambda\theta + (\lambda+\mu)\left(l\frac{\partial \theta}{\partial l} + \theta\right) - \\ &-(\lambda+\mu)\sum_1^\infty{}_m \frac{1}{R_2^{2m+1} - R_1^{2m+1}}\left(\frac{R_2}{l}\right)^{2m+1}\left[\frac{(m+2)(2m+3)(R_2^2-R_1^2)l^{2m+1}}{R_2^{2m+3}-R_1^{2m+3}} + 2m^2 - m - 4\right](R_1^{2m+1}X_m - \\ &- l^{2m+1}X_{-(m+1)}) + (\lambda+\mu)\sum_1^\infty{}_m \frac{1}{R_2^{2m+1}-R_1^{2m+1}}\left[\frac{(m-1)(2m-1)(R_1R_2)^{2m-1}(R_2^2-R_1^2)}{(R_2^{2m-1}-R_1^{2m-1})l^{2m+1}} + \right. \\ &\left. - 2m^2 - 5m + 1\right](R_2^{2m+1}X_m - l^{2m+1}X_{-(m+1)})\Big\}.\end{aligned} \qquad (65)$$

The other two formulas relative to $\bar{\omega}_2$ and $\bar{\omega}_3$ are deduced from the written one by, as usual, having been expanded by cyclic permutation of $\mathfrak{L}, \mathfrak{M}, \mathfrak{N}; x, y, z$. Hence we divide by l, integrate, and note that the arbitrary function introduced by integration reduces to a constant, since it must be harmonic and regular in S and independent of l. Because

$$-\frac{1}{R_1}\int_{\sigma_1}\left(\frac{\partial \mathfrak{N}}{\partial y} - \frac{\partial \mathfrak{M}}{\partial z}\right)d\sigma + \frac{1}{R_2}\int_{\sigma_2}\left(\frac{\partial \mathfrak{N}}{\partial y} - \frac{\partial \mathfrak{M}}{\partial z}\right)d\sigma = \int_\sigma (zM - yN)\,d\sigma = 0\,(^*),$$

and the terms of zero degree in $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}_3$, are zero, a harmonic and regular

---

* It is therefore sufficient to note that

$$\int_\sigma (zM - yN)\,d\sigma = \int_\sigma (\mathfrak{M}\cos nz - \mathfrak{N}\cos ny)\,d\sigma =$$

$$= -\int_S \left(\frac{\partial \mathfrak{M}}{\partial z} - \frac{\partial \mathfrak{N}}{\partial y}\right)dS = -\Sigma_i \int_S \left(\frac{\partial \mathfrak{M}_i}{\partial z} - \frac{\partial \mathfrak{N}_i}{\partial y}\right)dS$$

and to perform the integration with respect to ρ, bearing in mind that, by virtue of the reciprocity theorem of the spherical functions, one term alone differs from zero.

function exists, with at least one arbitrary constant, in S of form

$$\int \frac{dl}{l}\left(\frac{\partial \mathfrak{N}}{\partial y} - \frac{\partial \mathfrak{M}}{\partial z}\right) \text{(*)}.$$

We may write /62

$$\left.\begin{aligned}
2\mu\varpi_1 = {} & \int\frac{dl}{l}\left(\frac{\partial\mathfrak{N}}{\partial y} - \frac{\partial\mathfrak{M}}{\partial z}\right) + \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\Big\{(2\lambda+\mu)\int\frac{dl}{l}\theta + (\lambda+\mu)\theta - \\
& -(\lambda+\mu)\sum_1^\infty{}_m \frac{1}{R_2^{2m+1} - R_1^{2m+1}}\left(\frac{R_2}{l}\right)^{2m+1}\left[\frac{(m+2)(2m+3)(R_2^2 - R_1^2)l^{2m+1}}{m(R_2^{2m+3} - R_1^{2m+3})} + \right. \\
& \left. - \frac{2m^2 - m - 4}{m+1}\right](R_2^{2m+1}X_m - l^{2m+1}X_{-(m+1)}) + \\
& -(\lambda+\mu)\sum_1^\infty{}_m \frac{1}{R_2^{2m+1} - R_1^{2m+1}}\left[\frac{(m-1)(2m-1)(R_1R_2)^{2m-1}(R_2^2 - R_1^2)}{(m+1)(R_2^{2m-1} - R_1^{2m-1})l^{2m+1}} - \right. \\
& \left. + \frac{2m^2 + 5m - 1}{m}\right](R_2^{2m+1}X_m - l^{2m+1}X_{-(m+1)})\Big\} + h_1, \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \quad (65')$$

indicating by h_1, h_2, h_3 three arbitrary constants. Let us also note that in the preceding formulas we may assume the terms of zero degree in θ to be zero, since θ in expression (65) appears only under derivative signs. We may thus consider that $\int\frac{dl}{l}\theta$ also represents a very definite function which is harmonic and regular in S.

The values of u, v, w still remain to be determined. We may therefore begin by noting that expressions (53') may be written:

$$\left.\begin{aligned}
l\frac{\partial u}{\partial l} = {} & \frac{\varrho}{2\mu} - \frac{\lambda}{2\mu}\theta x - \varpi_3 y + \varpi_2 z - \frac{\lambda+\mu}{2\mu}x\left(l\frac{\partial\theta}{\partial l} + \theta\right) + \frac{\lambda+\mu}{2\mu}xX_0 + \\
& + \frac{\lambda+\mu}{2\mu}x\sum_1^\infty{}_m\frac{(m+1)(2m+1)}{R_2^{2m+1} - R_1^{2m+1}}(R_2^{2m+1}X_m - l^{2m+1}X_{-(m+1)}) + \\
& + \frac{\lambda+\mu}{2\mu}x\sum_1^\infty{}_m\frac{2m+1}{R_2^{2m+1} - R_1^{2m+1}}\Big\{-2 + \\
& + \frac{(m+2)[(R_1R_2)^2(R_2^{2m+1} - R_1^{2m+1}) + l^{2m+3}(R_2^2 - R_1^2)]}{(R_2^{2m+3} - R_1^{2m+3})l^2}\Big\}\left(\frac{R_2}{l}\right)^{2m+1}(R_1^{2m+1}X_m - l^{2m+1}X_{-(m+1)}) + \\
& + \frac{\lambda+\mu}{2\mu}\sum_2^\infty{}_m\frac{m-1}{R_2^{2m+1} - R_1^{2m+1}}\Big[-l^2 + \\
& + \frac{l^{2m+1}(R_2^{2m+1} - R_1^{2m+1}) - l^2(R_1R_2)^{2m-1}(R_2^2 - R_1^2)}{(R_2^{2m-1} - R_1^{2m-1})l^{2m+1}}\Big]\frac{\partial}{\partial x}(R_2^{2m+1}X_m - l^{2m+1}X_{-(m+1)}) + \\
& + \frac{\lambda+\mu}{2\mu}\sum_1^\infty{}_m\frac{m+2}{R_2^{2m+1} - R_1^{2m+1}}\Big[l^2 - \\
& - \frac{(R_1R_2)^2(R_2^{2m+1} - R_1^{2m+1}) + l^{2m+3}(R_2^2 - R_1^2)}{R_2^{2m+3} - R_1^{2m+3}}\Big]\left(\frac{R_2}{l}\right)^{2m+1}\frac{\partial}{\partial x}(R_1^{2m+1}X_m - l^{2m+1}X_{-(m+1)}). \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \quad (53''')$$

* In the contrary case, as results when we consider the expansion of any function harmonic in S into a series of spherical functions, integration introduces a term of form $\int\frac{dl}{l} = \log l$ which is not harmonic.

from which by dividing by l and integrating, we derive /63

$$
\left.\begin{aligned}
2\mu u = {} & \int \frac{dl}{l}\varrho - \lambda\frac{x}{l}\int \theta\, dl - 2\mu\left(\frac{y}{l}\int \varpi_3\, dl - \frac{z}{l}\int \varpi_2\, dl\right) - \\
& - (\lambda+\mu)x\theta + (\lambda+\mu)xX_0 + \\
& + (\lambda+\mu)x\sum_1^\infty {}_m \frac{2m+1}{R_2^{2m+1}-R_1^{2m+1}}\left(R_2^{2m+1}X_m - l^{2m+1}X_{-(m+1)}\right) + \\
& + (\lambda+\mu)x\sum_1^\infty {}_m \frac{2m+1}{R_2^{2m+1}-R_1^{2m+1}}\left[\frac{2}{m} + \right. \\
& \left. + \frac{(m+2)l^{2m+3}(R_2^2-R_1^2)-(m+1)(R_1R_2)^2(R_2^{2m+1}-R_1^{2m+1})}{(m+1)(R_2^{2m+3}-R_1^{2m+3})l^2}\right]\left(\frac{R_2}{l}\right)^{2m+1}\left(R_1^{2m+1}X_m - \right. \\
& \left. - l^{2m+1}X_{-(m+1)}\right) + (\lambda+\mu)\sum_2^\infty {}_m \frac{m-1}{R_2^{2m+1}-R_1^{2m+1}}\left[-\frac{l^2}{m+1} + \right. \\
& \left. + \frac{ml^{2m+1}(R_2^{2m+1}-R_1^{2m+1})+(m-1)l^2(R_1R_2)^{2m-1}(R_2^2-R_1^2)}{m(m-1)(R_2^{2m-1}-R_1^{2m-1})l^{2m+1}}\right]\frac{\partial}{\partial x}\left(R_2^{2m+1}X_m - \right. \\
& \left. - l^{2m+1}X_{-(m+1)}\right) + (\lambda+\mu)\sum_1^\infty {}_m \frac{m+2}{R_2^{2m+1}-R_1^{2m+1}}\left[-\frac{l^2}{m} + \right. \\
& \left. + \frac{(m+1)(R_1R_2)^2(R_2^{2m+1}-R_1^{2m+1})-(m+2)l^{2m+3}(R_2^2-R_1^2)}{(m+1)(m+2)(R_2^{2m+3}-R_1^{2m+3})}\right]\left(\frac{R_2}{l}\right)^{2m+1}\frac{\partial}{\partial x}\left(R_1^{2m+1}X_m - \right. \\
& \left. - l^{2m+1}X_{-(m+1)}\right) + k_1,
\end{aligned}\right\} \quad (66)
$$

. . . .

Let us now make the following observations concerning these formulas. Since

$$
-\frac{1}{R_1}\int_{\sigma_1}\varrho\, d\sigma + \frac{1}{R_2}\int_{\sigma_2}\varrho\, d\sigma = \int_\sigma L\, d\sigma = 0
$$

we may consider that $\int\frac{dl}{l}\varrho$ represents a function which is harmonic and regular in S and very determinate. Since the terms of degree -1 in θ; $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}_3$ are zero, the expressions $\frac{1}{l}\int \theta dl$; $\frac{1}{l}\int \bar{\omega}_1\, dl$, ... will represent harmonic functions as above; $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}_3$ contain the arbitrary constants h_1, h_2, h_3. Finally, k_1, k_2, k_3, as biharmonic functions regular in S and independent of l, are arbitrary constants.

6. *Cases in Which Other Conditions Are Given on the Limiting Spheres.* Taking due advantage of formulas (48) and (66), we will be able, without great difficulty, to solve the problems in which the components of the displacements /64 along one of the coordinate axes and the stress components along the other two coordinate axes, or the displacement components along two coordinate axes and the stress components along the other coordinate axis, are given on the two spherical surfaces. It would therefore help us to bear in mind our procedure in report I to solve similar problems for the full sphere.

Based on procedures similar to the preceding ones, all those problems in which there are different conditions -- and conditions similar to those to which we have already alluded -- for the case of the body limited by two parallel planes, are given on the two spherical surfaces. We would then have to make use of the harmonic function which assumes given values on a spherical surface, while on the other spherical surface the normal derivative assumes assigned values.*

Finally, let us also allude to the problems in which the normal displacements and tangential stresses -- or vice versa, the normal stresses and the tangential displacements -- are given on two spherical surfaces. Let us also refer to the problems in which the normal displacements and tangential stresses are given on one spherical surface, and on the other -- the normal stresses and the tangential displacements. These problems which may be solved by bearing in mind the procedure which we followed to treat similar problems for the entire sphere in the last subsection of another publication (Memoria del Circolo matematicale di Palermo, Vol. 17, 1903).

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* Or, even more simply, we could continue to start with expressions (48) and follow the general method advanced by Thomson, which consists of methodically applying the obvious observation that any biharmonic function regular in S is the sum of a harmonic function which on the surface assumes the same values of the biharmonic function and of a biharmonic function which becomes zero on the surface. If, for example, the values of displacements on σ_1 and of stresses on σ_2 are given, the values of U_m, V_m, W_m are determined only in part. They will be completely determined by subjecting the biharmonic functions $\lambda\theta x + 2\mu(l\frac{\partial U}{\partial l} + \bar{\omega}_3 y - \bar{\omega}_2 z)$, ... to the assumption of given values on σ_2. To satisfy these conditions, the preceding biharmonic functions will be formulated, and the terms will be neglected which become zero on the surface in order to have a harmonic function . The problem will then be simplified and easily solved.